For each three dimensional object described below,

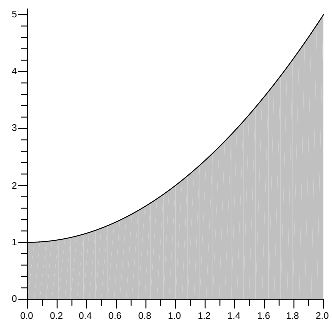
(a) Sketch the object

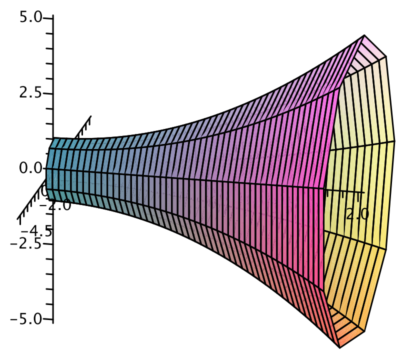
(b) Set up an integral that gives you the volume of the object

(c) Evaluate the integral to find the volume

1. The solid formed when the graph of $y = x^2 + 1$ from $x = 0$ to $x = 2$ is rotated about the x-axis.

(a)





(b)

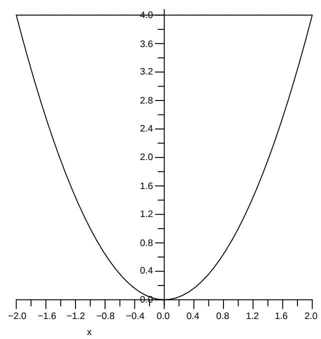
$$
\begin{aligned}
\text{Volume} &= \int_0^2 A(x)\,dx \\
&= \pi \int_0^2 \text{radius}^2\,dx \\
&= \pi \int_0^2 (x^2+1)^2\,dx
\end{aligned}
$$

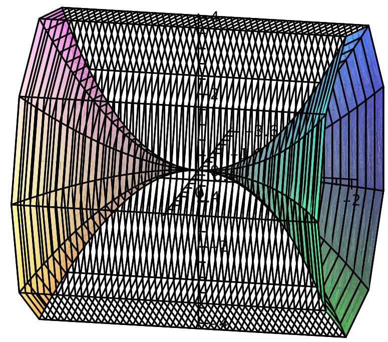
(c)

$$
\begin{aligned}
\text{Volume} &= \pi \int_0^2 x^4 + 2x^2 + 1\,dx \\
&= \pi\left(\frac{x^5}{5} + \frac{2x^3}{3} + x\right) \text{ from 0 to 2} \\
&= \pi\left[\left(\frac{32}{5} + \frac{16}{3} + 2\right) - 0\right] \\
&= \pi\left(\frac{96+80+30}{15}\right) \\
&= \frac{206\pi}{15}
\end{aligned}
$$

2. The solid formed when the region bounded by $y = x^2$ and $y = 4$ is rotated about the x-axis.

(a)





(b)

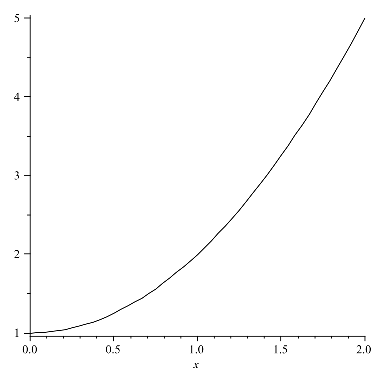
$$\begin{aligned}
\text{Volume} &= 2 \cdot \int_0^2 A(x)\,dx \\
&= 2\pi \int_0^2 (\text{outer radius})^2 - (\text{inner radius})^2\,dx \\
&= 2\pi \int_0^2 (4)^2 - (x^2)^2\,dx \\
&= 2\pi \int_0^2 16 - x^4\,dx
\end{aligned}$$

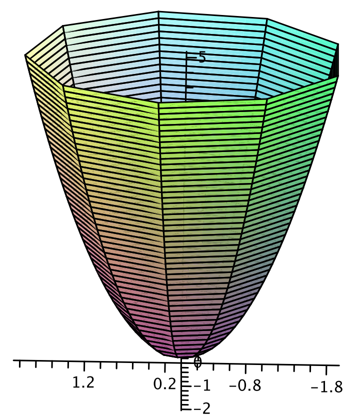
(c)

$$\begin{aligned}
\text{Volme} &= 2\pi(16x - \frac{x^5}{5}) \text{ from 0 to 2} \\
&= 2\pi[(32 - \frac{32}{5}) - 0] \\
&= 2\pi(\frac{128}{5}) \\
&= \frac{256\pi}{5}
\end{aligned}$$

3. Repeat #1 and #2, but rotating the region about the y-axis rather than the x-axis. (In the case of #2, only rotate the region bounded by $y = x^2$ and $y = 4$ that lies in the first quadrant.)

   (a) $y = x^2 + 1$ from $x = 0$ to $x = 2$, rotated about the y axis.
   Because we were only told to rotate the curve $y = x^2 + 1$ and find the enclosed volume, notice that even though we're rotating the same curve as we did in (1), we're enclosing a totally different volume:

      i. 



      ii. This time, our cross-sections are going to be horizontal circles, rather than vertical. How big or small a cross-section is will depend on where along the y-axis it is. In other words, the radius will be $x = g(y)$, rather than $y = f(x)$ as we've seen in the first two. And of course, this will in turn mean that we'll be integrating with respect to y rather than to x. (You can think of dy as the thickness of a slice. If you sketch in such a slice, you'll see you'd measure its thickness up and down – that is, in the y direction.)
      So we need to rewrite $y = x^2 + 1$ as a function of y. Solving for x, we find that

$$x = \sqrt{y - 1}.$$

Also, instead of talking about the left-most slice and the right-most slice, which gives us $x = a$ and $x = b$, we need to figure out where the bottom-most and top-most slices area, which will give us $y = c$ and $y = d$.
Looking at the picture, you can see the bottom-most slice occurs at the bottom of the parabola, which is at $y = 1$. The top-most slice occurs at the top of the parabola. When $x = \pm 2$, $y = 5$. Thus we'll integrate from $y = 1$ to $y = 5$.

$$\begin{aligned} \text{Volume} &= \int_1^5 A(y)\, dy \\ &= \pi \int_1^5 \text{radius}^2\, dy \\ &= \pi \int_1^5 (\sqrt{y-1})^2\, dy \\ &= \pi \int_1^5 y - 1\, dy \end{aligned}$$

iii.

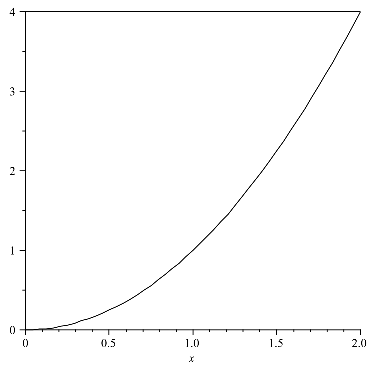
$$\begin{aligned} \text{Volume} &= \pi \left(\frac{y^2}{2} - y \right) \Big|_1^5 \\ &= \pi \left[\left(\frac{25}{2} - 5 \right) - \left(\frac{1}{2} - 1 \right) \right] \\ &= 8\pi \end{aligned}$$

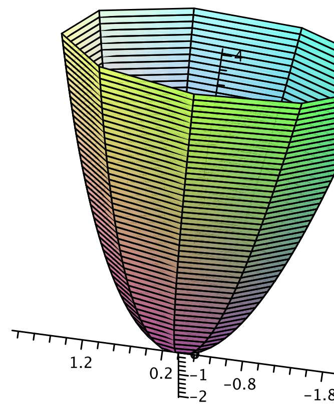
(b) $y = x^2$ and $y = 4$, rotated about the y-axis. Only rotate the region in the first quadrant.

First of all, the reason I only wanted you to rotate the region in the first quadrant is because if you rotate the whole region, you only need to rotate by 180 degrees to form a solid with circular cross-sections – if you rotate the full 360 degrees around the y-axis, you'd just be overlapping what you've already formed.

What that means as far as volume calculations is that you'd get double the volume ... but it *also* is harder to do the whole region, because you'd have to do two functions. So ... you'd work harder to get the wrong answer. All 'round, something to be avoided.

i.





ii. We need to find the radius of our cross-sections. As in the first part of this problem, since we're rotating around the y-axis, the radius of a cross-section will depend on how high up or down it is, or in other words, we need the radius to be a function of y. We therefore solve $y = f(x)$ for x, so that we can have the radius is $x = g(y)$.
Solving $y = x^2$ for x, we find that the radius is $x = \sqrt{y}$ from $y = 0$ to $y = 4$

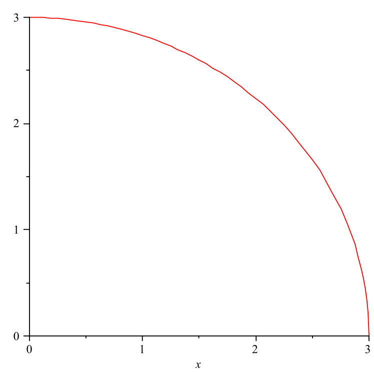
$$\begin{aligned}
\text{Volume} &= \int_0^4 A(y)\,dy \\
&= \pi \int_0^4 \text{radius}^2\,dy \\
&= \pi \int_0^4 (\sqrt{y})^2\,dy \\
&= \pi \int_0^4 y\,dy
\end{aligned}$$

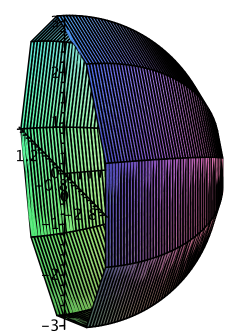
iii.

$$\begin{aligned}\text{Volume} &= \pi(\frac{y^2}{2}) \text{ from 0 to 4} \\ &= \pi[(\frac{16}{2}) - (0)] \\ &= \pi(8)\end{aligned}$$

4. The sphere of radius r.

   (a) The sphere of radius r can be formed by rotating the upper half of a circle of radius r about the x-axis. In fact, we can just rotate the upper right quarter-circle, and multiply the resulting volume by 2.





   (b) A circle of radius r, centered at the origin, has equation $x^2 + y^2 = r^2$, so the upper half has equation $y = \sqrt{r^2 - x^2}$.

$$\begin{aligned}V &= 2\left[\pi \int_0^r (\sqrt{r^2 - x^2})^2 \, dx\right] \\ &= 2\pi \int_0^r r^2 - x^2 \, dx\end{aligned}$$

(c)

$$
\begin{aligned}
V &= 2\pi r^2 x - \frac{x^3}{3}\Big|_0^r \\
&= 2\pi[(r^3 - \frac{r^3}{3}) - (0)] \\
&= \frac{4\pi r^3}{3}
\end{aligned}
$$