1. Find the following derivatives. Don't worry about algebraic simplifications.

   (a) $\frac{d}{dx}\left(\arcsin\left(\frac{2x}{e^x}\right)\right)$

   Since the function $\frac{2x}{e^x}$ is *inside* the function $\arcsin(x)$, I need to use the *chain rule.*

$$\begin{aligned}\frac{d}{dx}\left(\arcsin\left(\frac{2x}{e^x}\right)\right) &= \frac{1}{\sqrt{1-\left(\frac{2x}{e^x}\right)^2}} \cdot \frac{d}{dx}\left(\frac{2x}{e^x}\right) \\ &= \frac{1}{\sqrt{1-\left(\frac{2x}{e^x}\right)^2}} \cdot \frac{e^x \cdot 2 - 2x \cdot e^x}{(e^x)^2} \\ &= \frac{1}{\sqrt{1-\left(\frac{2x}{e^x}\right)^2}} \cdot \frac{2-2x}{e^x}\end{aligned}$$

   (b) $\frac{d}{dx}(x^2 \arctan(\ln(x))$

   x^2 is multiplied by the arctangent function *and* $\ln(x)$ is *inside* the arctangent function. Thus I need to use both the product rule and the chain rule:

$$\begin{aligned}\frac{d}{dx}(x^2 \arctan(\ln(x)) &= x^2 \cdot \frac{d}{dx}(\arctan(\ln(x)) + \frac{d}{dx}(x^2) \cdot \arctan(\ln(x)) \\ &= x^2 \cdot \frac{1}{1+(\ln(x))^2} \cdot \frac{d}{dx}(\ln(x)) + 2x \cdot \arctan(\ln(x)) \\ &= x^2 \cdot \frac{1}{1+(\ln(x))^2} \cdot \frac{1}{x} + 2x \arctan(\ln(x)) \\ &= \frac{x}{1+(\ln(x))^2} + 2x \arctan(\ln(x))\end{aligned}$$

   (c) $\frac{d}{dx}(\arctan(\arcsin(x)))$

Chain rule!

$$\begin{aligned}\frac{d}{dx}(\arctan(\arcsin(x))) &= \frac{1}{1+(\arcsin(x))^2}\cdot\frac{d\arcsin(x)}{dx}\\ &= \frac{1}{1+(\arcsin(x))^2}\cdot\frac{1}{\sqrt{1-x^2}}\\ &= \frac{1}{(1+(\arcsin(x))^2)\sqrt{1-x^2}}\end{aligned}$$

2. For each of the following, find an antiderivative.

(a) $g(x) = \dfrac{3}{x^2}$

First, I rewrite $g(x)$ as $3x^{-2}$, so that's written as a power of x. Then I just add one to the power, and divide by that new power.

$$G(x) = 3\cdot\frac{x^{-1}}{-1} = -\frac{3}{x}.$$

(b) $h(x) = \dfrac{3}{1+x^2}$

This I recognize as being related to the $\arctan(x)$.

$$h(x) = 3\cdot\frac{1}{1+x^2} \Longrightarrow H(x) = 3\cdot\arctan(x).$$

(c) $k(x) = \dfrac{7}{\sqrt{1-(x/6)^2}}$

I recognize that I've got a $\sqrt{1-u^2}$ present in the denominator, and just a constant in the numerator. I thus conclude that the antiderivative probably has something to do with arcsin.

We know that $7\cdot\dfrac{d}{du}(\arcsin(u)) = \dfrac{7}{\sqrt{1-u^2}}$. Thus since what I have is $\dfrac{7}{\sqrt{1-(x/6)^2}}$, it must have come from $7\arcsin(x/6)$ in some way.

Differentiating this would use the chainrule, giving an additional factor of $1/6$. To compensate for that, I need to divide by $1/6$, or multiply by 6:

$$k(x) = 7 \cdot \frac{1}{\sqrt{1-(x/6)^2}} \Longrightarrow K(x) = 7 \cdot 6 \cdot \arcsin(x/6).$$

Check:

$$K'(x) = 7 \cdot 6 \cdot \frac{1}{\sqrt{1-(x/6)^2}} \cdot \frac{1}{6} = \frac{7}{\sqrt{1-(x/6)^2}} = k(x),$$

as desired.

3. For each of the following, find the signed area given by the integral shown.

   (a) $\int_0^1 \frac{1}{4\sqrt{1-x^2}}\,dx$

   Using the FTC, I know I just need to antidifferentiate the integrand and plug in the limits of integration:

$$\begin{aligned} \int_0^1 \frac{1}{4\sqrt{1-x^2}}\,dx &= \left[\frac{1}{4}\arcsin(x)\right]_0^1 \\ &= \frac{1}{4}(\arcsin(1) - \arcsin(0)) \\ &= \frac{1}{4}(\pi/2 - 0) \\ &= \pi/8 \end{aligned}$$

   (b) $\int_{-\pi/2}^{\pi/3} \cos(-3x)\,dx$

   As Problem 2(c), where I needed to compensate for the multiplicative factor of 1/6 inside the square root, I need to compensate for the multiplicative factor of -3 inside the cosine function by dividing by -3 outside:

$$\begin{aligned}\int_{-\pi/2}^{\pi/3} \cos(-3x)\, dx &= -\frac{1}{3}\left[\sin(-3x)\right]_{-\pi/2}^{\pi/3} \\ &= -\frac{1}{3}(\sin(-\pi) - \sin(3\pi/2)) \\ &= -\frac{1}{3}(0 - (-1)) \\ &= -\frac{1}{3}\end{aligned}$$

(c) $\int_0^1 \frac{1}{1+9x^2}\, dx$

First of all, I notice that

$$\int_0^1 \frac{1}{1+9x^2}\, dx = \int_0^1 \frac{1}{1+(3x)^2}\, dx.$$

I am thus in a very similar situation to that of Problem 2(c), except that I'm dealing with the arctangent rather than the arcsine.

Once again, I need to compensate for the additional factor of 3 by dividing by 3:

$$\int_0^1 \frac{1}{1+(3x)^2}\, dx = \frac{1}{3}\left[\arctan(3x)\right]_0^1 = \frac{1}{3}(\arctan(3) - \arctan(0)) = \frac{\arctan(3)}{3}$$

(d) $\int_1^{e^{\pi/4}} \frac{\sin(\ln(x))}{x}\, dx$

This one is considerably harder to do simply by a guess and check method. Since we know from the FTC that the easiest way to approach this problem is to try to antidifferentiate the integrand, we have to think of something that differentiates to $\frac{\sin(\ln(x))}{x}$.

Looking closer, we see that we have one function ($\ln(x)$) inside another (sin). This indicates to us that our integrand must have come from differentiating something using the chain rule.

Since the chain rule tells us, in part, to differentiate the outside function leaving the inside function the same, we can conclude that the original function had to have involved (again, in part)

$$-\cos(\ln(x)).$$

Before we try to figure out any more, let's just check to see what differentiating this gives us:

$$\begin{aligned}\frac{d}{dx}(-\cos(\ln(x))) &= \text{ using the chain rule } \sin(\ln(x))\cdot\frac{d}{dx}(\ln(x)) \\ &= \sin(\ln(x))\cdot\frac{1}{x} \\ &= \frac{\sin(\ln(x))}{x}.\end{aligned}$$

Lo and behold, this was actually what we were trying to antidifferentiate in this first place.
Thus

$$\begin{aligned}\int_1^{e^{\pi/4}} \frac{\sin(\ln(x))}{x}\,dx &= [-\cos(\ln(x))]_1^{e^{\pi/4}} \\ &= -\cos(\ln(e^{\pi/4}) + \cos(\ln(1)) \\ &= -\cos(\pi/4) + \cos(0) \\ &= -\frac{\sqrt{2}}{2} + 1.\end{aligned}$$

Note: If you find this guessing and checking method very hard to follow, you are *not* alone. It was to surely avoid just such pain and agony that various methods of integration were developed. We'll learn two in this class – substitution and integration by parts.