Find the following integrals, and *check your answers!!*

1. $\displaystyle\int \frac{1}{\sqrt{1-x}}\,dx \qquad (u = 1-x)$

   If we let $u = 1-x$, then by differentiating u, we find that $\frac{du}{dx} = -1$ so $du = -1\,dx$ so $dx = -1\,du$. Replacing $1-x$ by u and dx by $-1\,du$ in the original integral, we find

$$\begin{aligned}\int \frac{1}{\sqrt{1-x}}\,dx &= \int \frac{1}{\sqrt{u}} \cdot -1\,du \\ &= -\int u^{-1/2}\,du \\ &= -\frac{1}{1/2}u^{1/2} + C \\ &= -2\sqrt{u} + C \\ &= -2\sqrt{1-x} + C\end{aligned}$$

   Check: Is $\frac{d}{dx}(-2\sqrt{1-x}+C) = \frac{1}{\sqrt{1-x}}$?

$$\frac{d}{dx}(-2\sqrt{1-x}+C) = -2\cdot\frac{1}{2}(1-x)^{-1/2}\cdot(-1)+0 = -1\cdot-1\cdot\frac{1}{\sqrt{1-x}} = \frac{1}{\sqrt{1-x}},$$

   as desired.

2. $\displaystyle\int x\sin(\pi x^2)\,dx \qquad (u = \pi x^2)$

   If we let $u = \pi x^2$, then differentiating u, we find that $\frac{du}{dx} = 2\pi x$. That means that $du = 2\pi x\,dx$, or $x\,dx = \frac{1}{2\pi}\,du$.

In the original integral, we replace πx^2 with u and $x\ dx$ with $\frac{1}{2\pi}\ du$:

$$\begin{aligned}\int x\sin(\pi x^2)\ dx &= \int \sin(\pi x^2)\cdot x\ dx \\ &= \int \sin(u)\cdot \frac{1}{2\pi}\ du \\ &= \frac{1}{2\pi}\int \sin(u)\ du \\ &= \frac{1}{2\pi}\cdot -\cos(u)+C \\ &= -\frac{1}{2\pi}\cos(\pi x^2)+C\end{aligned}$$

Check: Is $\frac{d}{dx}(-\frac{1}{2\pi}\cos(\pi x^2)+C) = x\sin(\pi x^2)$?

$$\frac{d}{dx}(-\frac{1}{2\pi}\cos(\pi x^2)+C) = -\frac{1}{2\pi}\cdot -sin(\pi x^2)\cdot 2\pi x = x\sin(\pi x^2),$$

as desired.

3. $\displaystyle\int_1^3 \frac{x}{1+x^2}\ dx \qquad (u = 1+x^2)$

If we let $u = 1+x^2$, then differentiating u gives us $\frac{du}{dx} = 2x$, so $du = 2x\ dx$, or $x\ dx = \frac{1}{2}\ du$.

Substituting these into the original integral, we get:

$$\begin{aligned}
\int_1^3 \frac{x}{1+x^2}\,dx &= \int_{x=1}^{x=3} \frac{1}{1+x^2}\cdot x\,dx \\
&= \int_{x=1}^{x=3} \frac{1}{u}\cdot\frac{1}{2}\,du \\
&= \frac{1}{2}\int_{x=1}^{x=3} \frac{1}{u}\,du \\
&= \frac{1}{2}\left[\ln(u)\right]_{x=1}^{x=3} \\
&= \frac{1}{2}\left[\ln(1+x^2)\right]_1^3 \\
&= \frac{1}{2}(\ln(10)-\ln(2)) \\
&= \frac{1}{2}\ln\left(\frac{10}{2}\right) \\
&= \frac{1}{2}\ln(5) \\
&= \ln(5^{1/2}) \\
&= \ln(\sqrt{5})
\end{aligned}$$

4. $\displaystyle\int \frac{x}{1+x^4}\,dx \qquad (u = x^2)$

If we let $u = x^2$, then differentiating u gives us $\dfrac{du}{dx} = 2x$, or $du = 2x\,dx$, or in other words, $x\,dx = \dfrac{1}{2}\,du$.

Substituting these into the original integral, we get:

$$\begin{aligned}
\int \frac{x}{1+x^4}\,dx &= \int \frac{1}{1+(x^2)^2}\cdot x\,dx \\
&= \int \frac{1}{1+u^2}\cdot\frac{1}{2}\,du \\
&= \frac{1}{2}\arctan(u) + C \\
&= \frac{1}{2}\arctan(x^2) + C
\end{aligned}$$

As usual, we can check this by differentiating our results. I'll leave that to you.

5. $\displaystyle\int_2^5 \frac{1}{x\ln(x)}\,dx \qquad (u = \ln(x))$

If we let $u = \ln(x)$, then differentiating u gives us $\dfrac{du}{dx} = \dfrac{1}{x}$. Thus $du = \dfrac{1}{x}\,dx$.

Substituting these into the original integral, we get:

$$\begin{aligned}
\int_2^5 \frac{1}{x\ln(x)}\,dx &= \int_{x=2}^{x=5} \frac{1}{\ln(x)} \cdot \frac{1}{x}\,dx \\
&= \int_{x=2}^{x=5} \frac{1}{u}\,du \\
&= [\ln(u)]_{x=2}^{x=5} \\
&= [\ln(\ln(x))]_2^5 \\
&= \ln(\ln(5)) - \ln(\ln(2))
\end{aligned}$$