1.1 $\int \arctan(x)\, dx$

Let

$$u = \arctan(x) \quad dv = dx$$
$$du = \frac{1}{1+x^2}\, dx \quad v = x$$

Thus

$$\int \arctan(x)\, dx = x\arctan(x) - \int \frac{x}{1+x^2}\, dx.$$

With a substitution, find that

$$\int \arctan(x)\, dx = x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C$$

Verify:

$$\frac{d}{dx}\left(x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C\right) = \frac{x}{1+x^2} + \arctan(x) - \frac{1}{2}\frac{2x}{1+x^2}(2x).$$

1.2 $\int e^x \cos(x)\, dx$

$$u = e^x \qquad dv = \cos(x)\, dx$$
$$du = e^x\, dx \qquad v = \sin(x)$$

$$\int e^x \cos(x)\, dx = e^x \sin(x) - \int e^x \sin(x)\, dx$$

$$u = e^x \qquad dv = \sin(x)\, dx$$
$$du = e^x\, dx \qquad v = -\cos(x)$$

$$\int e^x \cos(x)\, dx = e^x \sin(x) - \left(e^x \cdot (-\cos(x)) + \int \cos(x) e^x\, dx \right)$$
$$= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\, dx$$

$$2 \int e^x \cos(x)\, dx = e^x \sin(x) + e^x \cos(x)$$

$$\int e^x \cos(x)\, dx = \frac{e^x}{2} (\sin(x) + \cos(x)) + C$$

1.3 $\int (\ln(x))^2 \, dx$

Method 1:

$u = (\ln(x))^2 \qquad dv = dx$

$du = 2\ln(x)/x \qquad v = x$

$$\int (\ln(x))^2 \, dx = x(\ln(x))^2 - \int 2\ln(x) \, dx$$

Use integration by parts again.

Method 2:

$u = \ln(x) \qquad dv = \ln(x) \, dx$

$du = \frac{1}{x} \, dx \qquad v = \int \ln(x) \, dx$

Use integration by parts

Either way, end up with

$$\int (\ln(x))^2 \, dx = x(\ln(x))^2 - 2x\ln(x) + 2x + C$$

1.4 $\int x^3 e^{x^2}\ dx$

If try:

$$u = x^3 \quad dv = e^{x^2}\ dx$$
$$du = 3x^2 \quad v = ????$$

If try:

$$u = e^{x^2} \quad dv = x^3\ dx$$
$$du = 2xe^{x^2}\ dx \quad v = \frac{x^4}{4}$$

$$\int x^3 e^{x^2}\ dx = \frac{1}{4}x^4 e^{x^2} - \frac{1}{2}\int x^4 e^{x^2}\ dx.$$

Just made life harder.

1.4 (continued): $\int x^3 e^{x^2}\, dx$

Try

$u = x^2$ $\qquad dv = xe^{x^2}\, dx$

$du = 2x\, dx$ $\qquad v = \int xe^{x^2}\, dx$

Let $w = x^2$, so $dw = 2x\, dx$

$v = \frac{1}{2}e^{x^2}$

Hence

$$\begin{aligned}\int x^3 e^{x^2}\, dx &= \frac{x^2}{2}e^{x^2} - \frac{1}{2}\int e^{x^2}\cdot 2x\, dx \\ &= \frac{x^2}{2}e^{x^2} - \int xe^{x^2}\, dx \\ &= \frac{x^2}{2}e^{x^2} - \frac{1}{2}e^{x^2} + C\end{aligned}$$

Verify:

$$\frac{d}{dx}\left(\frac{x^2}{2}e^{x^2} - \frac{1}{2}e^{x^2} + C\right) = \left(\frac{x^2}{2} * 2x * e^{x^2} + e^{x^2} * x\right) - xe^{x^2} = x^3 e^{x^2}$$

1. 5 $\int x^2 \arcsin(x)\, dx$

Let

$u = \arcsin(x) \qquad dv = x^2\, dx$

$du = \frac{1}{\sqrt{1-x^2}}\, dx \qquad v = \frac{1}{3}x^3$

$$\int x^2 \arcsin(x) = \frac{1}{3}x^3 \arcsin(x) - \frac{1}{3}\int x^3 \frac{1}{\sqrt{1-x^2}}\, dx.$$

Let

$u = x^2 \qquad dv = \frac{x}{\sqrt{1-x^2}}\, dx$

$du = 2x\, dx \qquad v = \int x(1-x^2)^{-1/2}\, dx$

Let $w = 1 - x^2$, so $dw = -2x\, dx$

$v = -\frac{1}{2}\int w^{-1/2}\, dw = -\frac{1}{2}\cdot\frac{2}{1}w^{1/2} = -(1-x^2)^{1/2}$

$$\int x^2 \arcsin(x)\, dx = \frac{1}{3}x^3 \arcsin(x) - \frac{1}{3}\left[-x^2\sqrt{1-x^2} + 2\int x\sqrt{1-x^2}\, dx\right]$$

1.6 $\int \sin(\ln(x))\,dx$

Let

$u = \sin(\ln(x)) \qquad dv = \ dx$

$du = \cos(\ln(x)) \cdot \frac{1}{x}\ dx \qquad v = x$

$$\int \sin(\ln(x))\,dx = x\sin(\ln(x)) - \int (\cos(\ln(x))\,dx$$

Let

$u = \cos(\ln(x))\,dx \qquad dv = \ dx$

$du = -\sin(\ln(x)) \cdot \frac{1}{x}\ dx \qquad v = x$

$$\begin{aligned}
\int \sin(\ln(x))\,dx &= x\sin(\ln(x)) - \left[x\cos(\ln(x)) + \int \sin(\ln(x))\,dx\right] \\
2\int \sin(\ln(x))\,dx &= x\sin(\ln(x)) - x\cos(\ln(x)) \\
\int \sin(\ln(x))\,dx &= \frac{1}{2}(x\sin(\ln(x)) - x\cos(\ln(x))) + C
\end{aligned}$$

2.1 $\displaystyle\int_1^{e^3} \frac{\ln(x)}{x}\, dx$

Substitution: If we write this as a product, $\ln(x) \cdot \frac{1}{x}$, we can see that the two pieces are related.
Let $u = \ln(x)$, so $du = \frac{1}{x}\, dx$.

2.2 $\displaystyle\int_1^{e^3} \frac{\ln(x)}{x^2}\, dx$

Parts: Written as a product, $\ln(x) \cdot \frac{1}{x^2}$, the two pieces are unrelated.
Let $u = \ln(x)$ and $dv = \frac{1}{x^2}\, dx$. Then $du = \frac{1}{x}$ and $v = -\frac{1}{x}$.

2.3 $\displaystyle\int x^3 \cos(x^4)\, dx$

Substitution: We have a composition, $\cos(x^4)$. Furthermore, we more or less have the derivative of the inside function.
Let $u = x^4$, so $du = 4x^3\, dx$.

2.4 $\int x^3 \cos(x^2)\, dx$

Parts and substitution: Product of unrelated pieces, so integration by parts. However, we also have a non-trivial composition, so substitution will have to come in as well: In order to antidifferentiate $\cos(x^2)$, we'll need an attached factor of x.
Let $u = x^2$ and $dv = x\cos(x^2)\, dx$.

2.5 $\int \cos(x^{1/3})\, dx$

Parts and substitution: We have composition, so begin with $u = x^{1/3}$, so $du = \frac{1}{3}x^{-2/3}\, dx \Rightarrow 3x^{2/3}\, du = dx \Rightarrow 3u^2\, du = dx$.
Make the substitution, use integration by parts.

2.6 $\int \sec(x)\, dx$.

Substitution: Do the suggested multiplication, then let $u = \sec(x) + \tan(x)$.