Some Sample Problems for Midterm 3

Midterm 3 will cover 8.1 (again), 8.2, 8.3 (expanded), 8.4, and 8.5. As with the second exam, it is of course inevitable that material from many of the previous sections will also appear; this is particularly true of Section 8.3.

As usual, you may have one sheet of notes. The format is the same as always: handwritten (by you) notes on one-side of a standard 8 1/2 x 11 sheet of paper.

Also as usual, you may begin the exam as early as 12:30, but no matter when you begin, you must turn it in no later than 2:00. (As usual, if you have class until 1, talk to me or e-mail me).

The usual words of caution: The following problems neither are a practice test nor should replace other studying. The test may not cover exactly the same material as these problems cover, and the test problems will not be just like these. The idea is simply to give you additional problems to practice on. *The more problems you do, the better!*

You are responsible for all material covered in your reading, whether or not we covered it in class.

- The basics of sequences, and l'Hôpital's Rule, were on the last exam, but it is imperative that you understand them for this exam.
- Understand the nuances of increasing and decreasing sequences, bounded sequences, and when we know a limit exists.
- Have a "lexicon" in your head of standard examples of series whose convergence or divergence we know. This lexicon should include not only $\sum_k \frac{1}{k^p}$ for various values of p, but at least also geometric series.
- Understand when comparisons to determine convergence or divergence are useful and when they are not.
- Understand the difference between a sequence and a series.
- Know the two (or three) sequences associated with a series, and understand the difference between them.
- Be able to explain ***precisely*** what it means for a series $\sum_{k=1}^{\infty} a_k$ to converge to the value S. In your explanation, be sure to make the distinction between the *terms* of the series and the *partial sums* of the series. Be especially careful with your use of pronouns and antecedents. Give plenty of examples to illustrate your points.

- Be able to recognize a geometric series, know how to tell whether it converges, and know how to evaluate what it converges to, whether or not you begin adding with the 0th power.

- Suppose $\lim_{k\to\infty} a_k = 2$. Do we know anything about $\sum_{k=1}^{\infty} a_k$? How about if $\lim_{k\to\infty} a_k = 0$?

  Conversely, suppose $\sum_{k=1}^{\infty} a_k = 2$. Do we know anything about $\lim_{k\to\infty} a_k$? How about if $\sum_{k=1}^{\infty} a_k$ diverges?

- Can we usually find exactly what a series converges to? If not, specifically when can we find the exact number?

- Is the nth term test a test for convergence or for divergence? When *is* it a test for convergence?

- Understand the Integral and the Comparison tests for convergence. Know the hypotheses of each.

- Understand how to draw sketches which illustrate connections between series or partial sums and improper or proper integrals.

- Be able to look at a series and decide whether to use nth term test, Comparison Test, or Integral Test.

- Know how to use the Integral Test or the Comparison Test to approximate a convergent series within any given margin of error.

- Know how to use the Integral Test or Comparison Test to find upper and lower bounds for a convergent series.

- Understand the Alternating Series Test and its hypotheses.

- Know how to use the Alternating Series Test to approximate a convergent alternating series within any given margin of error, or to find upper and lower bounds of such a series.

- Understand why if we know $\sum |a_k|$ converges, then we can conclude that $\sum a_k$ also converges.

- Understand the Ratio Test and its hypotheses. This is our newest test, of course be sure to do as many practice problems with it as you can, to get comfortable with it.
- Be able to determine whether a series is divergent, conditionally convergent, or absolutely convergent (using not only the integral and comparison test, but also the ratio test).
- Be able to do all the assigned problems, and all those below

1. Show that $\sum_{j=17}^{\infty} \left(\frac{e}{\pi}\right)^j$ converges and find the ***exact*** value of the series (i.e. no decimal approximations).

2. Let $a(x)$ be a continuous function that is positive and decreasing for all $x \geq 1$. Let $a_k = a(k)$. Rank the following values. Draw diagrams to explain your answer.

$$A = \sum_{k=5}^{n-1} a_k \qquad B = \sum_{k=6}^{n} a_k \qquad C = a_5 + \int_6^n a(x)\,dx \qquad D = \int_5^n a(x)\,dx$$

3. Determine whether each of the following series converges or diverges. If the series converges, find upper and lower bounds for it.

   (a) $\sum_{m=1}^{\infty} \frac{m^3}{m^5+3}$

   (b) $\sum_{n=0}^{\infty} \frac{n!}{(2n)!}$
   Hint: This series *does* converge. After showing that, to find the upper bound, first show that $\frac{a_{n+1}}{a_n} \leq \frac{1}{2}$, then use this to compare each term in this sequence to a multiple of a_0 and a power of $\frac{1}{2}$, which will then allow you to compare this series to a larger geometric series.

4. Determine whether each of the following series converges or diverges. If the series converges, approximate it within 0.001. If the series diverges, find a value of N for which $S_N \geq 1000$.

(a) $\displaystyle\sum_{j=5}^{\infty} \frac{j^2}{1000j^2+2}$

(b) $\displaystyle\sum_{n=9}^{\infty} n^2 e^{-\frac{n^3}{1000}}$

5. Do the following series diverge, converge conditionally, or converge absolutely? If the series converges, find upper and lower bounds for it.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{\pi n - 1}$

(b) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{2k}{k^3+2}$

6. Do the following series diverge, converge conditionally, or converge absolutely? If the series converges, approximate it within 0.001.

(a) $\displaystyle\sum_{r=10}^{\infty} \frac{(-1)^r}{r-6}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-6)^k k^{10}}{k!}$

7. If $a_k > 0$ and $\displaystyle\sum_{k=0}^{\infty} a_k$ converges, prove that $\displaystyle\sum_{k=0}^{\infty} {a_k}^2$ converges as well.

8. (a) Prove that the every-other-term harmonic series $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots$ diverges.

   (b) Would the every-third-term harmonic series $1+\frac{1}{4}+\frac{1}{7}+\frac{1}{10}+\ldots$ diverge? How about the every-fourth-term harmonic series $1+\frac{1}{5}+\frac{1}{9}+\frac{1}{13}+\ldots$? Make as general a statement as possible about such series.

9. Determine whether $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{(2k-1)(2k-3)\cdots 5\cdot 3\cdot 1}$ converges or diverges.