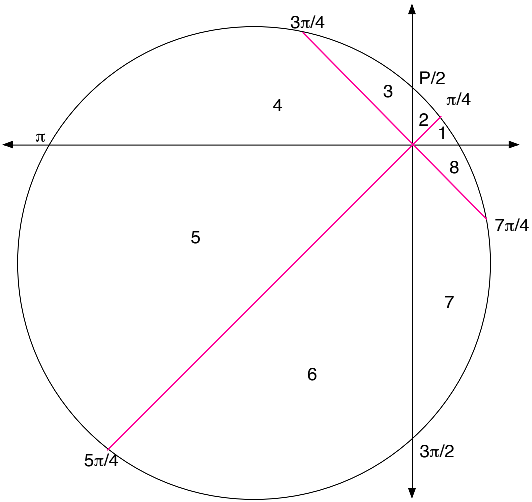
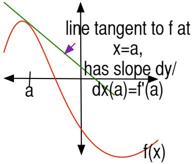
Horrible Pizza-Cutting:

Area of odd slices = Area of even slices

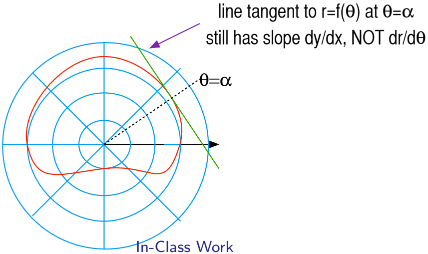


Calculus of Polar Functions: Derivatives

- If $y = f(x)$, then $f'(a)$ gives the instantaneous rate of change at $x = a$, and also the slope of the line tangent to $y = f(x)$ at $x = a$.



- If $r = f(\theta)$, then $f'(\alpha)$ still gives the instantaneous rate of change at $\theta = \alpha$, but it does **not** give the slope of the tangent line.



Calculus of Polar Functions: Derivatives

If $r = f(\theta)$, $\dfrac{dr}{d\theta}$ doesn't measure the slope of the line tangent to the graph:

- $\dfrac{dy}{dx}$ is a limit of slopes of secant lines, $\frac{\Delta y}{\Delta x}$.
- $\dfrac{\Delta r}{\Delta \theta}$ isn't the slope of a line

To find the slope of the tangent line, combine the conversion formulas $x = r\cos(\theta)$ and $y = r\sin(\theta)$ with the function$r = f(\theta)$, so that $x = f(\theta)\cos(\theta)$ and $y = f(\theta)\sin(\theta)$. Then use that $\left.\dfrac{dy}{dx}\right|_{\theta=\alpha} = \dfrac{\frac{dy}{d\theta}(\alpha)}{\frac{dx}{d\theta}(\alpha)}$.

Calculus of Polar Functions: Area

While we can calculate $\int_{\alpha}^{\beta} f(\theta)\, d\theta$, it no longer represents area:

- $\int_{a}^{b} f(x)\, dx$ is the limit of sums of areas of rectangles that lie more or less within our region, each of area $f(x)\Delta x$.
- $f(\theta)\Delta\theta$ doesn't measure the area of a rectangle that lies within our region.
- Our regions can have both top and bottom as the same curve. Even if we were going to do area, it wouldn't be area down to the x-axis
- It would be radial area.
- It turns out that radial area enclosed by $r = f(\theta)$ is given by

$$\text{Area} = \int_{\alpha}^{\beta} \frac{1}{2} r^2\, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} \left[f(\theta) \right]^2 d\theta.$$

Calculus of Polar Functions: Arclength

The formula for arc length is also different when working with polar functions.

- In rectangular coordinates, recall that the arc length of $y = f(x)$ from $x = a$ to $x = b$ is

$$\text{Arclength} = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx.$$

- For a polar function, the arc length of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ iis given by

$$\text{Arclength} = \int_\alpha^\beta \sqrt{\left[f(\theta)\right]^2 + \left[f'(\theta)\right]^2}\, d\theta$$

Calculus of Polar Functions: Signed Volume

Suppose $z = f(r, \theta)$ describes a surface.

Signed Volume is still given by $\iint_R f(r, \theta)\, dA$.

- In rectangular coordinates, partition R into rectangles of area $\Delta A = \Delta x \Delta y$. Fubini's theorem let's us break into iterated integrals $dx\, dy$ isn't totally surprising.
- Don't use rectangles to partition a polar region: use little bits of the polar grid. Those don't have area $\Delta A = \Delta r \Delta \theta$.

  Can use geometry to show: $\Delta A = r \Delta r \Delta \theta$.
- Fubini's Theorem for polar functions:

  Suppose that $f(r, \theta)$ is continuous on the region R: $\alpha \leq \theta \leq \beta$ and $g(\theta) \leq r \leq h(\theta)$ for all θ in $[\alpha, \beta]$. Then

$$\iint_R f(r, \theta)\, dA = \int_\alpha^\beta f(r, \theta) \cdot r\, dr\, d\theta.$$

WeBWorK Problem 10 - discussed last class

Find the volume below $z = 33 - x^2 - y^2$ and above $z = 8$. That is, find the volume enclosed by these two surfaces. The region we are thus integrating over is the circular intersection of the circular paraboloid with the horizontal plane.

$$
\begin{aligned}
V &= \iint_R 25 - x^2 - y^2 \, dA \\
&= \int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} 25 - x^2 - y^2 \, dy \, dx \\
&= \int_{-5}^{5} \left[25y - x^2 y - \frac{1}{3} y^3 \right]_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} dx \\
&= \int_{-5}^{5} \left(25\sqrt{25 - x^2} - x^2\sqrt{25 - x^2} - \frac{(25 - x^2)^{3/2}}{3} \right) \\
&\qquad - \left(-25\sqrt{25 - x^2} + x^2\sqrt{25 - x^2} + \frac{(25 - x^2)^{3/2}}{3} \right) dx
\end{aligned}
$$

WeBWorK Problem 10 - converting to polar coordinates

$$\begin{aligned} V &= \iint_R 25 - x^2 - y^2 \, dA \\ &= \int_{-5}^{5} \underbrace{50\sqrt{25-x^2}}_{\text{somewhat hard}} - \underbrace{2x^2\sqrt{25-x^2}}_{\text{hard}} - \underbrace{\frac{2}{3}(25-x^2)^{3/2}}_{\text{hard}} \, dx \end{aligned}$$

But if we convert to polar coordinates

(remember, $\iint_R f(r,\theta)\, dA = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r,\theta) r \, dr \, d\theta$)

$$\begin{aligned} V &= \iint_R 25 - r^2 \, dA = \int_0^{2\pi} \int_0^5 (25 - r^2) r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^5 25r - r^3 \, dr \, d\theta \end{aligned}$$