Solutions from last time

2. For $g(x,y) = x^2y + 4x + y^2 - 8y + xy + 20$, find all points where $g_x(x,y)$ and $g_y(x,y)$ are both zero.

Notice that if I just set the two partials equal to each other from the beginning, I'm trying to find **every** *point where they two partials are equal, and not using that they're in fact both 0.*

$$\begin{aligned} & g_x(x,y) = 0 \\ \Longrightarrow & 2xy + 4 + y = 0 \\ \Longrightarrow & (2x+1)y = -4 \\ \Longrightarrow & y = -\frac{4}{2x+1} \end{aligned}$$

$$\begin{aligned} & g_y(x,y) = 0 \\ \Longrightarrow & \; x^2 - 2y - 8 + x = 0 \\ \Longrightarrow & \; 2y = x^2 + x - 8 \end{aligned}$$

Solutions from last time

2. (continued) For $g(x,y) = x^2y + 4x + y^2 - 8y + xy + 20$, find all points where $g_x(x,y)$ and $g_y(x,y)$ are both zero.

$$g_x(x,y) = 0 \implies y = -\frac{4}{2x+1} \qquad g_y(x,y) = 0 \implies 2y = x^2 + x - 8.$$

$$\begin{aligned} g_x(x,y) = 0 = g_y(x,y) \quad &\implies \quad x^2 + x - 8 = 2y = 2\left(-\frac{4}{2x+1}\right) \\ &\implies \quad 2x^3 + x^2 + 2x^2 + x - 16x - 8 = -8 \\ &\implies \quad 2x^3 + 3x^2 - 15x = 0 \\ &\implies \quad x(2x^2 + 3x - 15) = 0 \\ &\implies \quad x = 0 \text{ or } x = \frac{-3 \pm \sqrt{9 - 120}}{4} \\ &\implies \quad x = 0 \\ &\implies \quad y = -4 \end{aligned}$$

Thus the only point where $g_x(x,y) = 0 = g_y(x,y)$ is the point $(0,-4)$.

Solutions from last time

2. (continued) **What is the significance of the point** $(0, -4)$ **on the graph of** $g(x, y) = x^2y + 4x + y^2 - 8y + xy + 20$**?**

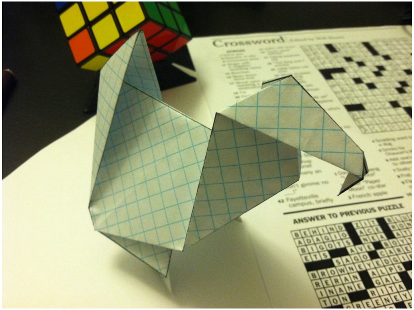
Solutions from last time

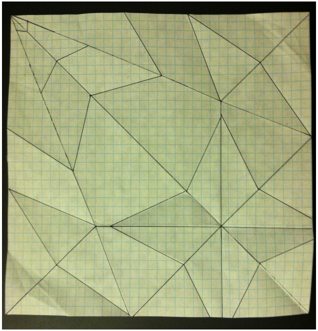
3. For $g(x,y) = x^2y + 4x + y^2 - 8y + xy + 20$, find g_{xy} and g_{yx}.
   Found in Problem 2, that $g_x(x,y) = 2xy + 4 + y$ and
   $g_y(x,y) = x^2 - 2y - 8 + x$
   Thus

$$g_{xy} = \frac{\partial}{\partial y}\left(2xy + 4 + y\right) = 2x + 1, g_{yx} = \frac{\partial}{\partial x}\left(x^2 - 2y - 8 + x\right) = 2x + 1$$

Fun Fact Friday

Origami!





Daily WW: $f(x,y) = 5\sin(2x+y) + 8\cos(x-y)$

A. $\frac{\partial f}{\partial x} = f_x = 5\cos(2x+y)(2\cdot 1+0) - 8\sin(x-y)(1-0) =$
$10\cos(2x+y) - 8\sin(x-y)$

B. $\frac{\partial f}{\partial y} = f_y = 5\cos(2x+y)(0+1) - 8\sin(x-y)(0-1) =$
$5\cos(2x+y) + 8\sin(x-y)$

C. $\frac{\partial^2 f}{\partial x^2} = f_{xx} = -10\sin(2x+y)(2\cdot 1+0) - 8\cos(x-y)(1-0) =$
$-20\sin(2x+y) - 8\cos(x-y)$

D. $\frac{\partial^2 f}{\partial y^2} = f_{yy} = -5\sin(2x+y)(0+1) + 8\cos(x-y)(0-1) =$
$-5\sin(2x+y) - 8\cos(x-y)$

E. $\frac{\partial^2}{\partial x \partial y} = f_{yx} = -5\sin(2x+y)(2\cdot 1+0) + 8\cos(x-y)(1-0) =$
$-10\sin(2x+y) + 8\cos(x-y)$

F. $\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = -10\sin(2x+y)(0+1) - 8\cos(x-y)(0-1) =$
$-10\sin(2x+y) + 8\cos(x-y)$

Recall:

Last time, you worked through the following problems:

2. Let $g(x, y) = x^2y + 4x - y^2 - 8y + xy + 20$. Find all points where $g_x(x, y)$ and $g_y(x, y)$ are both zero. What is the significance of these points?

Found: only such point is the point $(0, -4)$.

What *is* the significance of this point?

In Class Work

1. Find and classify (as best you can) all critical points of the function $h(x, y) = x^2 - 4x - 23 - y^3 - 9y^2 - 27y + xy$

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2. The **wave equation** is the partial differential equation $c^2 \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial t^2}$. Show that the functions $f_n(x, t) = \sin(n\pi x)\cos(n\pi ct)$ satisfy the wave equation, for any positive integer n and any constant c.

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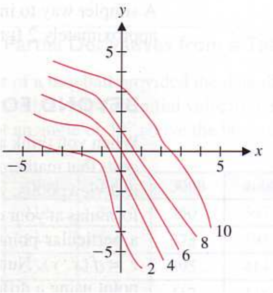
3. Use the contour plot at right to estimate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the origin.



Solutions

1. Find and classify (as best you can) all critical points of the function $h(x,y) = x^2 - 4x - 23 - y^3 - 9y^2 - 27y + xy$

$$h_x = 2x - 4 + y \qquad h_y = -3y^2 - 18y - 27 + x$$

$$h_x = 0 \Rightarrow x = 2 - y/2 \qquad h_y = 0 \Rightarrow -3y^2 - 18y - 27 + (2 - y/2) = 0$$

$$\begin{aligned}
\Rightarrow -3y^2 - 37y/2 - 25 &= 0 \\
\Rightarrow 6y^2 + 37y + 50 &= 0 \\
\Rightarrow y &= \frac{-37 \pm \sqrt{(37)^2 - 4(6)(50)}}{2(6)} \\
\Rightarrow y &= \frac{-37 \pm 13}{12} \\
\Rightarrow y &= -2 \text{ or } y = -\frac{25}{6}
\end{aligned}$$

Solutions

1. *(continued)* Find and classify (as best you can) all critical points of the function $h(x, y) = x^2 - 4x - 23 - y^3 - 9y^2 - 27y + xy$

We found that $h_x = 2x - 4 + y$ and $h_y = -3y^2 - 18y - 27 + x$.

When we solved $h_x = 0$ and $h_y = 0$, we found

$$y = -2 \qquad \text{or} \qquad y = -\frac{25}{6}.$$

Using $h_x = 0 \Rightarrow x = 2 - y/2$, the two critical points are $(3, -2)$ and $(\frac{49}{12}, -\frac{25}{6})$.

Solutions

1. *(continued)* Find and classify (as best you can) all critical points of the function $h(x, y) = x^2 - 4x - 23 - y^3 - 9y^2 - 27y + xy$

What can we figure out from the 2nd partials?

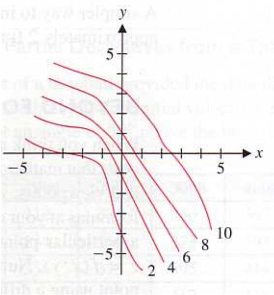
$$h_{xx} = 2 \qquad h_{yy} = -6y - 18.$$

At $(3, -2)$, $h_{xx}(3, -2) = 2 > 0 \Rightarrow h$ is concave up parallel to x; $h_{yy}(3, -2) = 0$ is uninformative.

At $(49/12, -25/6)$,$h_{xx}(49/12, -25/6) = 2 > 0 \Rightarrow h$ is concave up parallel to x; $h_{yy}(49/12, -25/6) = 7 > 0 \Rightarrow h$ is concave up parallel to y.

Thus we know that neither point is a local maximum (since in at least one direction, the surface is concave up in each case). They each could be minima or saddle points. Look at contour plot to decide.

2. Use the contour plot at right to estimate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the origin.



Key idea: $\frac{\partial f}{\partial x}(0,0) \approx \left.\frac{\Delta f}{\Delta x}\right|_{(0,0)}$, and that we know the value of f along the contour plots.

3. The **wave equation** is the partial differential equation $c^2 \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial t^2}$. Show that the functions $f_n(x, t) = \sin(n\pi x)\cos(n\pi ct)$ satisfy the wave equation, for any positive integer n and any constant c.

Key idea: $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial t^2}$, and then showing that multiplying $\frac{\partial^2 f}{\partial x^2}$ by c^2 will produce $\frac{\partial^2 f}{\partial t^2}$.

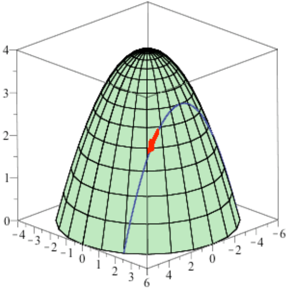
Preview of §12.4:

- If $f(x)$ is a differentiable function, we can use the tangent line at $x = x_0$,

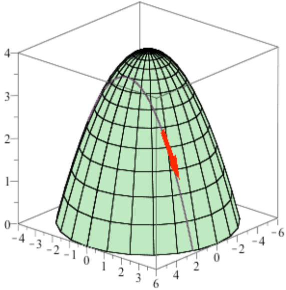
$$y = L(x) = f'(x_0)(x - x_0) + f(x_0)$$

as a **linear approximation** of f at x_0.

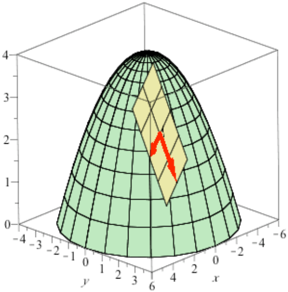
- In the same way, for a function $f(x, y)$, the *tangent plane* at (a, b) will give a linear approximation of f at (a, b).



Above, the red arrow represents the line to the curve formed by the intersection of the plane $y = b$ with the surface $z = f(x, y)$.



This arrow represents the tangent line to the curve at (a, b) when $x = a$ is fixed.



These two lines (and every other tangent line to a curve on the surface that goes through the point $(a, b, f(a, b))$ will lie on our **tangent plane**, shown in yellow.

Finding the tangent plane:

In order to find the equation of a plane, we need: **a point** and **a normal vector**.

- **Point:** use the point of tangency, $(a, b, f(a, b))$.
- **Normal vector:** Find it by first finding the direction vectors for each tangent line, then taking the cross product.