Group Problem Set 2

1. Find a symbolic expression for the general term a_k of the sequence

$$\frac{1}{2}, -\frac{1}{7}, \frac{1}{28}, -\frac{1}{63}, \frac{1}{126}, \ldots$$

Hint: You may want to look for a pattern in the types of numbers the denominators are near.

2. Here is one way to create the figure below: The largest square is drawn first, and is shaded black. A square whose side length is $r\%$ as long as the larger square is inscribed with one vertex on each edge of the larger square, and is shaded white. This process is then repeated recursively - that is, at each stage, we inscribe a square whose side length is $r\%$ as long as the previously-drawn square so that one vertex lies on each edge of the previous square, and we alternate shading the new square black or white.

(a) If we refer to the initial square as square 0, find a formula for the area A_k of the kth square in terms of the area of the previous square (that is, the $k-1$st square).

(b) The formula you found in part (a) defines the area of the kth square *recursively* – that is, each subsequent area is defined in terms of the previous one. Find an equivalent way to define the sequence $\{A_k\}_{k=0}^{\infty}$ that allows us to evaluate the area at any stage directly (each area will be in terms of r and the side length of the original, largest, square).

(c) If the original side length is 1, use your formula from part (b) to find the area of the square at stage 50 (that is, $k = 50$), in terms of r.

(d) Again assuming the original square has side length 1, what is the area of the shaded portion of this picture?

3. Let $a(x)$ be a continuous function that is positive and decreasing for all $x \geq 1$. Let $a_k = a(k)$. Rank the following values. Draw diagrams to explain your answer.

$$A = \sum_{k=5}^{n-1} a_k \qquad B = \sum_{k=6}^{n} a_k \qquad C = a_5 + \int_6^n a(x)\,dx \qquad D = \int_5^n a(x)\,dx$$

4. Determine whether each of the following series converges or diverges. If the series converges, find a partial sum S_N guaranteed to approximate the series within 0.001. (You don't need to find the value of the partial sum, just a reasonable value for N.)

(a) $\displaystyle\sum_{j=1}^{\infty} je^{-j}$

(b) $\displaystyle\sum_{j=5}^{\infty} \frac{j^2}{1000j^2+2}$

(c) $\displaystyle\sum_{m=1}^{\infty} \frac{m^3}{m^5+3}$

5. Let $\{a_k\}$ be a sequence, and (as is usual) let $S_N = \displaystyle\sum_{k=1}^{N} a_k$. If $S_N = 7 - \dfrac{1}{N^2}$,

(a) what is the value of $\displaystyle\sum_{k=1}^{10} a_k$?

(b) what is the value of $\displaystyle\sum_{k=1}^{3} a_k$?

(c) what is the value of $\displaystyle\sum_{k=4}^{10} a_k$?

(d) find an expression for a_4.

(e) find an expression for the general term a_k.

(f) does the series $\displaystyle\sum_{k=1}^{\infty} a_k$ converge? (Explain your answer, of course). If so, what does it converge to?

(g) find $\displaystyle\lim_{k\to\infty} a_k$.