Determine whether each series converges or diverges. If the series converges, find a number N such that the partial sum S_N approximates the sum of the series within .001. If the series diverges, find a number N such that $S_N \geq 1000$.

1. $\displaystyle\sum_{n=4}^{\infty} \frac{2n}{(n^2+5)^{2/3}}$

- **Determine Convergence or Divergence**
  - **Try the nth term test:**

    As has happened so often, the nth term test is inconclusive. (The bottom behaves more or less like $n^{4/3}$, which is (just barely) a higher power than the numerator. Since the denominator therefore increases more rapidly than the numerator, the limit is 0.)
  - **Try the Integral test:**

    After discovering that the nth term test isn't going to help me,I have to determine which test to use. I see that it's in a form that I'm somewhat comfortable developing a comparison from, but I also see that I've got $n^2 + 5$ in the denominator and $2n$ in the numerator, which makes it amenable to the integral test.

    Since the integral test is easier to develop approximations from, I'll use the integral test.

    > *Incidentally, if you do try to determine the convergence/divergence of this using the comparison test it ends up being more difficult than you might at first think.*

    * **Check that the integral test applies:**

      Is $f(x) = \dfrac{2x}{(x^2+5)^{2/3}}$ continuous, positive, and (eventually) decreasing? Continuous and positive, yes. But decreasing?

If I want to absolutely convince myself, I can take the derivative and make sure it's negative. So ...

$$\begin{aligned}
\frac{d}{dx}\left(\frac{2x}{(x^2+5)^{2/3}}\right) &= \frac{(x^2+5)^{2/3}\cdot 2 - 2x\cdot(2/3)(x^2+5)^{-1/3}(2x)}{(x^2+5)^{4/3}} \\
&= \frac{(6(x^2+5)-8x^2)/3(x^2+5)^{1/3}}{(x^2+5)^{4/3}} \\
&= \frac{-2x^2+30}{3(x^2+5)^{5/3}}
\end{aligned}$$

Since $3(x^2+5)^{5/3}$ is always positive, the sign of this derivative depends entirely upon the sign of the numerator. $-2x^2+30 \leq 0$ when $2x^2 \geq 30$, or $x^2 \geq 15$, or $x \geq 4$ or so. So this function does decrease monotonically, once we get past $x = 4$.

* **Apply the Integral Test :**

Since the integral test applies, we know that the series we're interested in will behave in the same way that the integral $\int_{x=4}^{\infty} \frac{2x}{(x^2+5)^{2/3}}\, dx$ does.

So . . . does $\int_{x=4}^{\infty} \frac{2x}{(x^2+5)^{2/3}}\, dx$ converge or diverge?

Use substitution:

$$\begin{aligned}
u &= x^2+5 \\
du &= 2x\, dx \\
\text{when } x = 4 & \quad u = 21 \\
\text{as } x \to \infty & \quad u \to \infty \text{ also.}
\end{aligned}$$

$$\int_4^{\infty} \frac{2x}{(x^2+5)^{2/3}}\, dx = \lim_{t\to\infty} \int_{21}^{t} \frac{1}{u^{2/3}}\, du,$$

which we know by the p-test diverges.

Therefore, by the integral test, our series also diverges, since the series does whatever the integral does.

- **Find a number N such that $S_N \geq 1000$.**

  How do I know such a number N even exists? Well, since

  $$\sum_{n=0}^{\infty} \frac{2n}{(n^2+5)^{2/3}}$$

  diverges (and since it's a positive term series), I know that if I add up the infinite number of terms, I'll get infinity. Since infinity is bigger than any finite number, that must mean that at some point in the adding process, I'm passing 1000, and then as I add in more terms, eventually I'll pass 1000000, etc. So if my series diverges, I can find a partial sum that's bigger than any number I might choose.

  How are we going to find a number N so that

  $$\sum_{n=4}^{N} \frac{2n}{(n^2+5)^{2/3}} \geq 1000?$$

  We can use an idea you developed on your homework. You found that as long as $a(x)$ is continuous, positive, and decreasing on $[1, \infty)$,

  $$\sum_{k=1}^{N} a_k \geq \int_1^{N+1} a(x)\, dx.$$

  Here, our lower limit is 4 rather than 1, but as long as our function is continuous, positive, and decreasing on $[4, \infty)$, that doesn't matter.

  So if we can find N so that $\displaystyle\int_4^{N+1} \frac{2x}{(x^2+5)^{2/3}} \geq 1000$, then we'll know that $S_N \geq 1000$ also.

$$\begin{aligned}
\int_4^{N+1} \frac{2x}{(x^2+5)^{2/3}}\,dx &\geq 1000 \\
\int_{x=4}^{x=N+1} u^{-2/3}\,du &\geq 1000 \\
3u^{1/3}\Big|_{x=4}^{x=N+1} &\geq 1000 \\
3(x^2+5)^{1/3}\Big|_4^{N+1} &\geq 1000 \\
3[(N+1)^2+5]^{1/3} - 3(21)^{1/3} &\geq 1000 \\
3[(N+1)^2+5]^{1/3} &\geq 1000 + 8.277 \\
((N+1)^2+5)^{1/3} &\geq \frac{1008.277}{3} \\
(N+1)^2+5 &\geq (336.09)^3 \\
(N+1)^2 &\geq (3.8\times 10^7) - 5 \\
N+1 &\geq \sqrt{3.8\times 10^7} \\
N &\geq 6161.46 - 1 \\
N &= 6161
\end{aligned}$$

Therefore $\displaystyle\sum_{n=4}^{6161} \frac{2n}{(n^2+5)^{2/3}} > 1000.$

Check using Maple:

```
> evalf(sum(2*n/(n^2+5)^(2/3),n=4..6161));

                                1000.523919
```

Sure enough, I've found a value of N that works.

You might be wondering how much bigger this value of N is than is really necessary. In other words, how much larger of an N did we produce by working with the smaller integral rather than with the actual sum. We can just experiment in Maple and (keeping in mind that there might be some approximation error), it seems that you have to add up to $N = 6157$ to actually go over 1000, so we're only adding 4 more terms than was absolutely necessary. Not bad!

2. $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k^6+17}$

- **Determine Convergence or Divergence:**
  - **Try the nth term test:**

    A single application of l'hopital's rule tells me that

    $$\lim_{k\to\infty} \frac{k}{k^6+17} = \lim_{k\to\infty} \frac{1}{6k^5} = 0.$$

    Thus once again the nth term test is inconclusive.
  - **Try the comparison test:**

    It's leaping out at me that

    $$\frac{k}{k^6+17} \leq \frac{k}{k^6} = \frac{1}{k^5}.$$

    Thus, I can compare the sums

    $$\sum_{k=0}^{\infty} \frac{k}{k^6+17} \leq \sum_{k=0}^{\infty} \frac{1}{k^5} \dots$$

    ... *except* that the series on the right will have a problem at $k = 0$. The series on the left doesn't have any problems at $k = 0$, however, so we deal with this by breaking the series on the left up:

    $$\sum_{k=0}^{\infty} \frac{k}{k^6+17} = \frac{0}{0^6+17} + \sum_{k=1}^{\infty} \frac{k}{k^6+17} = 0 + \sum_{k=1}^{\infty} \frac{k}{k^6+17}.$$

    Now we can say:

    $$\sum_{k=0}^{\infty} \frac{k}{k^6+17} \leq 0 + \sum_{k=1}^{\infty} \frac{1}{k^5} = \sum_{k=1}^{\infty} \frac{1}{k^5}.$$

    The series on the right converges because $p > 1$, so our smaller series also converges by the **comparison test**.

- **Find a number N so that the partial sum S_N is within .001 of the value of the series.**

  In other words, find a number N so that $\sum_{k=0}^{N} \frac{k}{k^6+17}$ is within .001 of $\sum_{k=0}^{\infty} \frac{k}{k^6+17}$.

  That is, we want to find N so that

  $$\sum_{k=0}^{\infty} \frac{k}{k^6+17} - \sum_{k=0}^{N} \frac{k}{k^6+17} \leq .001.$$

  Since

  $$\begin{aligned} \sum_{k=0}^{\infty} a_k - \sum_{k=0}^{N} a_k &= (a_0 + a_1 + \ldots + a_{N-1} + a_N + a_{N+1} + \ldots) \\ &\quad -(a_0 + a_1 + \ldots + a_{N-1} + a_N) \\ &= a_{N+1} + a_{N+2} + \ldots \\ &= \sum_{k=N+1}^{\infty} a_k \\ &= R_N, \end{aligned}$$

  we want to find N so that $R_N \leq .001$.

  Goal: Find N so that $R_N \leq .001$.

  I only know how to do this in a few limited cases – geometric series and with the integral test. I can't integrate my term, and it's not a geometric series. Where to go from here?

  Well, we know from the comparison we already did that

  $$R_N \leq \sum_{k=N+1}^{\infty} \frac{1}{k^5}.$$

  So if we make the sum on the right be sufficiently small, the remainder on the left will be as well.

  Use the integral test results on the sum on the right:

  $$R_n \leq \sum_{k=N+1}^{\infty} \frac{1}{k^5} \leq \int_N^{\infty} \frac{1}{x^5}\, dx.$$

$$\begin{aligned}\int_N^\infty \frac{1}{x^5}\,dx &= \lim_{t\to\infty}\left[-\frac{1}{4x^4}\right]_N^t \\ &= \lim_{t\to\infty} -\frac{1}{4t^4}+\frac{1}{4N^4} \\ &= \frac{1}{4N^4}\end{aligned}$$

Thus if we find N so that $\frac{1}{4N^4} \leq .001$, then $R_N \leq .001$.

$$\frac{1}{4N^4} \leq .001 \Rightarrow 4N^4 \geq 1000 \Rightarrow N^4 \geq 250 \Rightarrow N \geq 3.98.$$

Use $N = 4$. We just found that

$$\sum_{k=5}^{\infty} \frac{1}{k^5} \leq .001,$$

which means that our smaller remainder R_4 will also be less than .001.

From that, we can conclude that $\sum_{k=0}^{4} \frac{k}{k^6+17}$ is within .001 of $\sum_{k=0}^{\infty} \frac{k}{k^6+17}$.