1. Use the ratio test to decide the convergence or divergence of

   (a) $\displaystyle\sum_{k=12}^{\infty} \frac{10^k}{k!}$:

   This is a positive term series, so the ratio test applies. We want to see whether $L = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ is less than 1 or not.

$$L = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{10^{k+1}}{(k+1)!}}{\frac{10^k}{k!}}.$$

   Simplifying, we find that

$$L = \lim_{k\to\infty} \frac{10^{k+1}}{(k+1)!} \cdot \frac{k!}{10^k} = \lim_{k\to\infty} \frac{10^{k+1}}{10^k} \cdot \frac{k!}{(k+1)!} = \lim_{k\to\infty} \frac{10}{k+1} = 0.$$

   Since $L < 1$, the ratio test tells us this series **converges**!

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^{50}}$

   $\dfrac{2^n}{n^{50}} > 0$ for all $n \geq 1$, so the ratio test applies.
   Let

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)^{50}}}{\frac{2^n}{n^{50}}}.$$

   Again, we want to see whether or not $L < 1$. Simplifying, we find

$$L = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^{50}} \cdot \frac{n^{50}}{2^n} = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{n^{50}}{(n+1)^{50}} = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^{50} = 2.$$

   Since $L > 1$, the ratio test tells us that this series **diverges.**

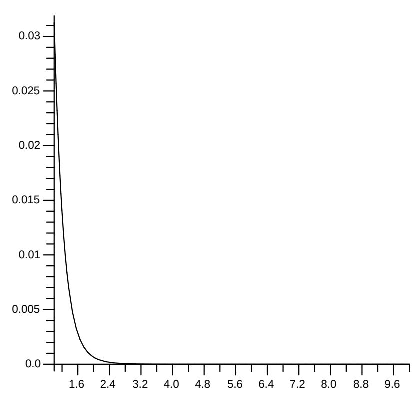
2. Use whatever test seems appropriate to determine the convergence or divergence of

   (a) $\sum_{m=1}^{\infty} \frac{m}{(1+m^2)^5}$

   - **nth term test:** Since $\lim_{m\to\infty} \frac{m}{(1+m^2)^5} = 0$, the nth term test is inconclusive.
   - **Integral test:** This series is screaming integral test to me, because if I let $u = 1 + x^2$, then $\frac{1}{2}\, du = x\, dx$.
     However, before I can use the integral test, I need to make sure it applies.

     Is $a(x) = \dfrac{x}{(1+x^2)^5}$ **continuous, positive and decreasing on** $[1, \infty)$**?**

     – On $[1, \infty)$, both the numerator and the denominator are positive, so $a(x)$ is positive.
     – On $[1, \infty)$, the denominator is never 0 and both the numerator and denominator are continuous. Thus $a(x)$ is continuous.
     – Is $a(x)$ decreasing on $a(x)$?
     While it's not a true proof, it's usually sufficient to look at a graph of $a(x)$ from 1 (or whatever your lower limit of integration is) to somewhere between 10 and 50.



It looks pretty clear that this function will continue to decrease, with the x-axis as a horizontal asymptote.

Note: If you have any doubts, and you want to show that $a(x)$ really is decreasing all the way out to infinity, then you need to take the derivative $a'(x)$, and show that it's always negative (or at any rate, is eventually negative).

Thus $a(x)$ **is** continuous, positive, and decreasing on $[1, \infty)$, and so the integral test applies.

The integral test tells me that my sum does whatever the corresponding improper integral does. That is

$$\sum_{m=1}^{\infty} \frac{m}{(1+m^2)^5} \text{ converges} \iff \int_1^{\infty} \frac{x}{(1+x^2)^5}\, dx \text{ converges.}$$

We'll use a substitution for this integral:

$$\begin{aligned} u &= 1+x^2 \\ \frac{1}{2}du &= x\,dx \end{aligned}$$

$$\begin{aligned} x = 1 &\Rightarrow u = 2 \\ x \to \infty &\Rightarrow u \to \infty \end{aligned}$$

$$\int_1^{\infty} \frac{x}{(1+x^2)^5}\, dx = \frac{1}{2}\int_2^{\infty} \frac{1}{u^5}\, du,$$

which we know converges, by the p-test.

So the integral, and hence the series $\displaystyle\sum_{m=1}^{\infty} \frac{m}{(1+m^2)^5}$, converges.

- **Comparison Test:** While I chose to use the integral test, the comparison test would have also worked on this problem; the easiest comparison to use would be

$$\frac{m}{(1+m^2)^5} \le \frac{m}{m^{10}} = \frac{1}{m^9}.$$

(b) $\displaystyle\sum_{j=1}^{\infty} \frac{1}{j+e^j}$

- **nth term test:** $\lim_{j\to\infty} \frac{1}{j+e^j} = 0$, so the nth term test is inconclusive.
- **Comparison Test:** While I wouldn't particularly care to integrate $\frac{1}{x+e^x}$, I have no problems coming up with a comparison, so I'm going to go with the comparison test.
  Next, I have to choose which of the two obvious comparisons to use.

$$\begin{aligned} j+e^j &\geq j, e^j \\ \Rightarrow \frac{1}{j+e^j} &\leq \frac{1}{j}, \frac{1}{e^j} \\ \Rightarrow \sum_{j=1}^{\infty} \frac{1}{j+e^j} &\leq \sum_{j=1}^{\infty} \frac{1}{j}, \sum_{j=1}^{\infty} \frac{1}{e^j} \end{aligned}$$

  Since $\sum_{j=1}^{\infty} \frac{1}{j}$ is the harmonic series, which we know diverges, and since being less than or equal to a divergent series is not useful, we won't use that comparison.

  On the other hand, $\sum_{j=1}^{\infty} \frac{1}{e^j}$ is a geometric series, with $r < 1$, so it converges. Since our series is less than this convergent series, this is a useful comparison.

  Thus $\displaystyle\sum_{j=1}^{\infty} \frac{1}{j+e^j}$ converges.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{n!}$

- **nth term test:** At this point, you may not be sure whether $n!$ is larger than n^3 or not. Perhaps we can figure that out later. So we'll skip the nth term test.
- **Ratio Test:**
  With the factorial in it, this one is screaming ratio test!

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\ &= \lim_{n\to\infty} \frac{(n+1)^3}{(n+1)!} \cdot \frac{n!}{n^3} \\ &= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \frac{1}{n+1} \\ &= 1 \cdot 0 = 0 \\ &< 1 \end{aligned}$$

Since $L < 1$, the series $\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{n!}$ converges.

- **Back to the nth term test:**
  What can we say about $\lim_{n\to\infty} \frac{n^3}{n!}$?
  From the nth term test, we know that *if* this limit turns out not to be zero, our series must diverge. But we just showed that our series converges. That has to mean that the limit of the nth term *is* zero.
  Conclusion: $\lim_{n\to\infty} \frac{n^3}{n!} = 0$
  In fact, $\lim_{n\to\infty} \frac{n^k}{n!} = 0$ for any $k > 0$.

(d) $\displaystyle\sum_{k=5}^{\infty} \frac{k^4 + 400k^3}{1000k^4 + k}$

- **nth term test:**

$$\lim_{k\to\infty} \frac{k^4 + 400k^3}{1000k^4 + k} = \frac{1}{1000} \neq 0.$$

  Therefore, while the sequence of terms converges to 1/1000, the series $\displaystyle\sum_{k=5}^{\infty} \frac{k^4 + 400k^3}{1000k^4 + k}$ diverges by the nth term test.