Let $I = \int_5^{10} \cos\left(\frac{x^2}{3}\right) + x \, dx$

1. Calculate L_{1000} and T_{1000}.

   I could just use the leftsum and trapezoid commands in Maple, but for practice, I'll actually figure these out.

   a. L_{1000}:
   $\Delta x = \frac{b-a}{n} = \frac{10-5}{1000} = \frac{1}{200}$.

$$\begin{aligned} L_{1000} &= \Delta x \sum_{i=0}^{999} f(a + i\Delta x) \\ &= \frac{1}{200} \sum_{i=0}^{999} f(5 + \frac{i}{200}) \\ &= \frac{1}{200} \sum_{i=0}^{999} \cos\left(\frac{(5+i/200)^2}{3}\right) + (5 + i/200) \end{aligned}$$

   Using Maple, it turns out that this is 37.35741781.

   $\boxed{\text{Thus if I use } L_{1000} \text{ to approximate } I\text{, I find that } I \approx 37.35741781.}$

   b. T_{1000}:
   As for the trapezoidal sum, the easiest way to figure it out is to use that it's the average of the right and left sums. Since right and left sums are so similar, I know that

$$\begin{aligned} R_{1000} &= \frac{1}{200} \sum_{i=1}^{1000} \cos\left(\frac{(5+i/200)^2}{3}\right) + (5 + i/200) \\ &= \text{(using Maple) } 37.38302527 \\ T_{1000} &= \frac{L_{1000} + R_{1000}}{2} \\ &= 37.38302527 \end{aligned}$$

   $\boxed{\text{Therefore, if I use } T_{1000} \text{ to approximate } I\text{, I find that } I \approx 37.38302527.}$

2. How close are these to the actual value of I?

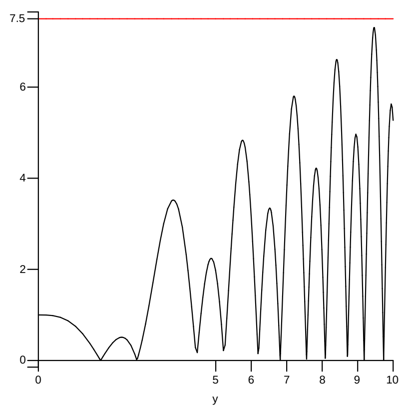
   $f(x) = \cos\left(\frac{x^2}{3}\right) + x$ on $[5, 10]$ is neither monotone nor always of the same concavity, so I can't use Thm 1 or Thm 2 to figure out a bound on the error. Theorem 3, however, *does* apply.

   a. Error in using L_{1000} to approximate I:
   From Theorem 3, I know that

   $$|I - L_{1000}| \leq \frac{K_1(b-a)^2}{2 \cdot 1000} = \frac{K_1(10-5)^2}{2000}.$$

   I find K_1 by graphing the absolute value of $f'(x)$ in Maple, over the interval $[5, 10]$. It turns out that 7.5 works quite nicely for K_1:

   Graph of $y = abs(f'(x))$ along with the horizontal line $y = 7.5$:



$$\begin{aligned} |I - L_{1000}| &\leq \frac{7.5(10-5)^2}{2000} \\ &\leq \frac{7.5(25)}{2000} \\ &\leq .09375 \\ &\leq .1 \end{aligned}$$

   So we can use 37.35741781 to approximate I, and know that the error in this approximation is less than .1.

   $\boxed{\text{In other words, } I = 37.35741781 \pm .1.}$

b. Error in using T_{1000} to approximate I:

From Theorem 3, I know that

$$|I - T_{1000}| \leq \frac{K_2(b-a)^3}{12 \cdot 1000^2} = \frac{K_2(10-5)^3}{12 \times 10^6}.$$

I find K_2 by graphing the absolute value of $f''(x)$ in Maple, over the interval $[5, 10]$. It turns out that 43 works well for K_2.

$$\begin{aligned} |I - T_{1000}| &\leq \frac{43(10-5)^3}{1.2 \times 10^7} \\ &\leq \frac{43(125)}{1.2 \times 10^7} \\ &\leq .00044792 \\ &\leq .00045 \end{aligned}$$

So we can use 37.38302527 to approximate I, and know that the error in this approximation is less than .00045.

$\boxed{\text{In other words, } I = 37.38302527 \pm .00045.}$

3. Find a value of n so that L_n approximates I accurate within 0.01.

We *want* $|I - L_n| \leq .01$.

We *know* $|I - L_n| \leq \dfrac{K_1(b-a)^2}{2n}$.

Goal: If we can find n so that $\dfrac{K_1(b-a)^2}{2n} \leq .01$, then we will have the string of inequalities

$$|I - L_n| \leq \frac{K_1(b-a)^2}{2n} \leq .01,$$

and so we will have that $|I - L_n| \leq .01$.

Therefore:

Need to find: n so that $\frac{K_1(b-a)^2}{2n} \leq .01$. Since $a = 5$, $b = 10$, and we've chosen $K_1 = 7.5$, this gives us

$$\begin{aligned}
\frac{7.5 \cdot (10-5)^2}{2n} &\leq .01 \\
\frac{7.5 \cdot 25}{2n} &\leq \frac{1}{100} \\
\frac{15 \cdot 25}{4n} &\leq \frac{1}{100} \\
\frac{15 \cdot 25 \cdot 100}{4} &\leq n \\
15 \cdot 25 \cdot 25 &\leq n \\
15 \cdot 625 &\leq n \\
9375 &\leq n
\end{aligned}$$

Therefore L_{9375} is guaranteed to be within 0.01 of the actual value of I.

(While I didn't ask you to do it, you could then go to Maple and find that L_{9375} is 37.3688623, so we now know that

$$I = 37.3688623 \pm .01.)$$