1. $\int xe^x \, dx$

The integrand of $\displaystyle\int xe^x \, dx$ is a product. There are no compositions involved, so it does not seem like a candidate for substitution. That leaves us with integration by parts as our first choice. It it ends up not getting us anywhere, we can always rethink our stance on substitution ...

Integration by parts:

$$\int u \, dv = uv - \int v \, du.$$

We need to choose u and dv so that $\int u \, dv = \int xe^x \, dx$. Looking at the right side, we're going to end up integrating the product of $v \, du$. We therefore want $v \, du$ to be somehow *easier* to antidifferentiate than xe^x is.

That means that we want to choose dv so that v isn't more "complicated" than dv, and we want to choose u so that du is "simpler" than u – keeping in mind all the while that the product of u and dv must equal $xe^x \, dx$.

Let's try:

$u = x$	$dv = e^x \, dx$
$du = dx$	$v = e^x$

This seems to satisfy our requirements: du is 1, which is less complicated than x (it's a lower power of x, for instance), and v is the same as dv, so it's no more complicated. Does it lead to something useful? Let's see!

Then using the above formula for integration by parts, we end up with

$$\int xe^x \, dx = xe^x - \int e^x \, dx = xe^x - e^x + C$$

Sure enough, rewriting our original integral using the integration by parts formula led to an integral that was a piece of cake to antidifferentiate!

We should of course always check by differentiation to make sure we're right:

Verify:

$$\frac{d}{dx}(xe^x - e^x + C) = xe^x + e^x - e^x = xe^x$$

Question: What would have happened had we chosen otherwise? Suppose we'd tried:

$$u = e^x \qquad dv = x\,dx$$
$$du = e^x\,dx \qquad v = \frac{x^2}{2}$$

Then using the formula for integration by parts, we end up with

$$\int xe^x\,dx = \frac{x^2}{2}e^x - \int \frac{x^2}{2}e^x\,dx.$$

The new integral on the right is even worse (in that we have $x^2/2$ rather than x) than the one we started out with! But it's still a product – could we use integration by parts again? Let's try it! We again have two "obvious" choices for u and dv – what happens?

- If we choose $u = \frac{x^2}{2}$ and $dv = e^x\,dx$, then $du = x\,dx$ and $v = e^x$, so we end up with

  $$\int xe^x\,dx = \frac{x^2}{2}e^x - [\frac{x^2}{2}e^x - \int xe^x\,dx] = \int xe^x\,dx.$$

  A true statement, but not particularly helpful!

- If instead we choose $u = e^x$ and $dv = \frac{x^2}{2}\,dx$, then $du = e^x\,dx$ and $v = \frac{x^3}{6}$, so we end up with

  $$\int xe^x\,dx = \frac{x^2}{2}e^x - [\frac{x^3}{6}e^x - \int \frac{x^3}{6}e^x\,dx].$$

  The integral we're left with became still more complicated!

We could try integration by parts again, but the writing is on the wall – either we'll end up where we started, or the integral with get even more complicated.

Thus in this case, the "other" choice for u and dv led to true statement that was not in any way useful.

Note: Just because integrating by parts a second time didn't work here doesn't mean it never works! See #4 below!

2. $\int x \ln(x)\, dx$

Let's try making the analogous choices as worked in the first problem:

$u = x \qquad\qquad dv = \ln(x)\, dx$

$du = \dfrac{x^2}{2}\, dx \qquad\qquad v = ???$

Once again we realize that even though we know a lot about integration, there are still some basic things we don't know! We haven't seen yet whether $\ln(x)$ has a nice antiderivative, and if so, what it is. (Just wait! That's number 5!)

So instead, let's try the other choices for u and dv:

$u = \ln(x) \qquad\qquad dv = x\, dx$

$du = \dfrac{1}{x}\, dx \qquad\qquad v = \dfrac{x^2}{2}$

It seems as if this might lead to the same problem as we saw in the first problem, but the fact that du is $\frac{1}{x}$ might help. Let's just see what happens:

$$\begin{aligned}
\int x \ln(x)\, dx &= uv - \int v\, du \\
&= \frac{x^2}{2}\ln(x) - \int \frac{x^2}{2} * \frac{1}{x}\, dx \\
&= \frac{x^2 \ln(x)}{2} - \frac{1}{2}\int x\, dx \\
&= \frac{x^2 \ln(x)}{2} - \frac{1}{2} * \frac{x^2}{2} + C
\end{aligned}$$

Verify:

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^2\ln(x)}{2} - \frac{x^2}{4} + C\right) &= \frac{1}{2} * x^2 * \frac{1}{x} + \frac{1}{2}\ln(x) * 2x - \frac{1}{4} * 2x \\ &= \frac{x}{2} + x\ln(x) - \frac{x}{2} \\ &= x\ln(x)\end{aligned}$$

3. $\displaystyle\int x\sec^2(x)\,dx$

The integrand $x\sec^2(x)$ is a product. If we think of it as a product of x and $\sec^2(x)$, the two parts of the product are unrelated. Our first thought, therefore, is to use integration by parts.

Using the formula $\displaystyle\int u\,dv = uv - \int v\,du$, with

$$\begin{array}{ll} u = x & dv = \sec^2(x)\,dx \\ du = dx & v = \tan(x) \end{array}$$

I get

$$\int x\sec^2(x)\,dx = x\tan(x) - \int \tan(x)\,dx.$$

Even though $\tan(x)$ is a basic "building-block" function, at first glance this looks like something we haven't learned how to do yet. Then we realize we can re-write

$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$

We're then looking at the integral $\displaystyle\int \frac{\sin(x)}{\cos(x)}\,dx$. If we let $u = \cos(x)$, then $-du = \sin(x)\,dx$, so we have

$$\begin{aligned}\int x\sec^2(x)\,dx &= x\tan(x) + \int \frac{1}{u}\,du \\ &= x\tan(x) + \ln|\cos(x)| + C\end{aligned}$$

Verify:

$$\begin{aligned}\frac{d}{dx}(x\tan(x)+\ln|\cos(x)|+C) &= [x\cdot\sec^2(x)+1\cdot\tan(x)]+\frac{1}{\cos(x)}\cdot -\sin(x)\\ &= x\sec^2(x)+\tan(x)-\tan(x)\\ &= x\sec^2(x)\end{aligned}$$

4. $\displaystyle\int x^2\cos(2x)\,dx$

The integrand is again a product. While the term $\cos(2x)$ *is* a composition (which might make us think of substitution), the inside of the composition – $2x$ – does not differentiate into the other term in the product. So we again turn first to integration by parts to see if we can make it work.

$$\begin{aligned} u &= x^2 & dv &= \cos(2x)\,dx\\ du &= 2x\,dx & v &= \frac{1}{2}\sin(2x)\end{aligned}$$

$$\begin{aligned}\int x^2\cos(2x)\,dx &= uv-\int v\,du\\ &= \frac{x^2}{2}\sin(2x)-\int x\sin(2x)\,dx\end{aligned}$$

Unfortunately, we still have an integral that's not a basic one. Does that mean that integration by parts has failed us? Not necessarily! We see that whereas before we had x^2 multiplied by a trig function, the degree has gone down and now we have x multiplied by a trig function. The integral we're left with, $\int x\sin(2x)\,dx$, is again a product of two unrelated terms. Let's see if another application of integration by parts on the remaining integral helps us!

$$\begin{aligned} u &= x & dv &= \sin(2x)\,dx\\ du &= dx & v &= -\tfrac{1}{2}\cos(2x)\end{aligned}$$

$$\begin{aligned}\int x^2\cos(2x)\,dx &= \frac{x^2}{2}\sin(2x) - [uv - \int v\,du] \\ &= \frac{x^2}{2}\sin(2x) - [-\frac{x}{2}\cos(2x) + \frac{1}{2}\int \cos(2x)\,dx] \\ &= \frac{x^2}{2}\sin(2x) + \frac{x}{2}\cos(2x) - \frac{1}{2} * \frac{1}{2}\sin(2x) + C \\ &= (\frac{x^2}{2} - \frac{1}{4})\sin(2x) + \frac{x}{2}\cos(2x) + C\end{aligned}$$

Verify:

$$\begin{aligned}\frac{d}{dx}((\frac{x^2}{2} - \frac{1}{4})\sin(2x) + \frac{x}{2}\cos(2x) + C) &= [(\frac{x^2}{2} - \frac{1}{4})(2\cos(2x)) + (x)\sin(2x)] \\ &\quad +[\frac{x}{2}(-2\sin(2x) + \frac{1}{2}\cos(2x)] \\ &= (x^2 - \frac{1}{2} + \frac{1}{2})\cos(2x) + (x-x)\sin(2x) \\ &= x^2\cos(2x)\end{aligned}$$

5. $\int \ln(x)\,dx$

This turns out to be one of the cooler applications of integration by parts. You look at this, and you think "this is a basic building-block function and I *ought* to know how to antidifferentiate it, but I don't." When you were working with $\tan(x)$ on the exam, rewriting the function in an equivalent way helped, but it's not obvious how to do that here. So what to do? There's no obvious choice of u to make for a substitution problem, as there's no composition going on. You might also say that there's no obvious multiplication going on – which is of course necessary for integration by parts – but you *always* have multiplication by 1 going on. It doesn't seem likely to go anywhere, but it's worth a try, since it's the only idea we've got!

Using the formula:

$$\int u\,dv = uv - \int v\,du,$$

with

$$u = \ln(x) \qquad dv = dx$$
$$du = \frac{1}{x}\,dx \qquad v = x$$

I get

$$\int \ln(x)\,dx = x\ln(x) - \int x \cdot \frac{1}{x}\,dx = x\ln(x) - \int 1\,dx = x\ln(x) - x + C$$

Lo and behold, fabulous cancellation occurred, and we found that integration by parts worked even though the product we tried seemed like we were grasping at straws!

Verify:

$$\frac{d}{dx}(x\ln(x) - x + C) = [x \cdot \frac{1}{x} + 1 \cdot \ln(x)] - 1 = 1 + \ln(x) - 1 = \ln(x).$$

6. $\int x^3 e^{x^2}\,dx$

Your first choice might be to try $u = x^3$ and $dv = e^{x^2}$. But that gets stalled right away, as you can't antidifferentiate e^{x^2}.

Your next choice might be to try $u = e^{x^2}$ and $dv = x^3$. But then $du = 2xe^{x^2}$ while $v = \frac{1}{4}x^4$, and when you look at your new integral, the power of x is even higher, so that's no good.

And then you're stuck, because it seems as if these are the only two choices. But *why* does it seem that way? What, after all, does x^3 mean? It's a product too, isn't it? We can write x^3 as $x \cdot x^2$ or even as $x \cdot x \cdot x$. Once you see that, it opens up tons of new choices.

Try this:

$$u = x^2 \qquad dv = xe^{x^2}\,dx$$
$$du = 2x\,dx \qquad v = ?$$

I need to find v, using substitution.

Let $w = x^2$. Then $dw = 2x\,dx$, and so $\frac{1}{2}\,dw = x\,dx$. Therefore $v = \int xe^{x^2}\,dx = \frac{1}{2}e^w\,dw = \frac{1}{2}e^w = \frac{1}{2}e^{x^2}$.

Thus I have

$$u = x^2 \qquad dv = xe^{x^2}\ dx$$
$$du = 2x\ dx \qquad v = \frac{1}{2}e^{x^2}$$

Hence

$$\begin{aligned}\int x^3 e^{x^2}\ dx &= \frac{x^2}{2}e^{x^2} - \frac{1}{2}\int e^{x^2} * 2x\ dx \\ &= \frac{x^2}{2}e^{x^2} - \int xe^{x^2}\ dx \\ &= \frac{x^2}{2}e^{x^2} - \frac{1}{2}e^{x^2} + C\end{aligned}$$

Verify:

$$\frac{d}{dx}\left(\frac{x^2}{2}e^{x^2} - \frac{1}{2}e^{x^2} + C\right) = \left(\frac{x^2}{2} * 2x * e^{x^2} + e^{x^2} * x\right) - xe^{x^2} = x^3 e^{x^2}$$

7. $\int e^x \cos(x)\ dx$

Usually, we try to choose our u to be something that gets "less complicated" when we differentiate it, while choosing for our dv something that doesn't get any more "complicated" when we antidifferentiate it. Your book as a pneumonic to help you remember the usual way of choosing things. But here, both portions of our product stay the same level of difficulty whether we differentiate or antidifferentiate. That seems to imply that it doesn't really matter which we choose as u. Let's just choose $u = e^x$.

$$u = e^x \qquad dv = \cos(x)\ dx$$
$$du = e^x\ dx \qquad v = \sin(x)$$

$$\int e^x \cos(x)\ dx = e^x \sin(x) - \int e^x \sin(x)\ dx$$

Hmmm, our new integral doesn't seem to be any better than our original. But it's not any worse either. What happens if we try again?

$$u = e^x \qquad dv = \sin(x)\ dx$$
$$du = e^x\ dx \qquad v = -\cos(x)$$

$$\begin{aligned}\int e^x \cos(x)\ dx &= e^x \sin(x) - \int e^x \sin(x)\ dx \\ &= e^x \sin(x) - (e^x * (-\cos(x)) + \int \cos(x) e^x\ dx) \\ &= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\ dx\end{aligned}$$

At first, we feel like we're going around in circles! But then, we notice that the integral on the right is the same as the integral on the left. An integral is just a mathematical expression like any other, and can be added or subtracted to both sides. So we add $\int e^x \cos(x)\ dx$ to both sides of the equation:

$$\begin{aligned}2\int e^x \cos(x)\ dx &= e^x \sin(x) + e^x \cos(x) \\ \int e^x \cos(x)\ dx &= \frac{e^x}{2}(\sin(x) + \cos(x)) + C\end{aligned}$$

Verify:

$$\frac{d}{dx}(\frac{e^x}{2}(\sin(x)+\cos(x))+C) = \frac{e^x}{2}(\cos(x)-\sin(x))+\frac{e^x}{2}(\sin(x)+\cos(x)) = e^x \cos(x)$$

8. $\int \arctan(x)\ dx$

   This is another building block function, and similarly to $\ln(x)$, there's no obvious way to rewrite it. So we might as well try the same technique that worked for $\ln(x)$!

$$\begin{aligned} u &= \arctan(x) & dv &= dx \\ du &= \frac{1}{1+x^2}\ dx & v &= x \end{aligned}$$

$$\begin{aligned}
\int \arctan(x)\,dx &= x\arctan(x) - \int \frac{x}{1+x^2}\,dx \\
&\qquad \text{let } u = 1+x^2\text{, so } du = 2x\,dx \Rightarrow \frac{1}{2}\,du = dx \\
&= x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C
\end{aligned}$$