1. $\displaystyle\int_1^{e^3} \frac{\ln(x)}{x}\,dx$

When we write the integrand as a product

$$\frac{\ln(x)}{x} = \ln(x)\cdot\frac{1}{x},$$

we see that the product is of a function and its derivative. That is, the two pieces of the product are directly related by differentiation. In that case, we first try substitution:

Substitution:

Let $u = \ln(x)$. Then $du = \frac{1}{x}\,dx$. Substituting in, we get

$$\begin{aligned}
\int_1^{e^3} \frac{\ln(x)}{x}\,dx &= \int_{x=1}^{x=e^3} u\,du \\
&= \left[\frac{1}{2}u^2\right]_{x=1}^{x=e^3} \\
&= \left[\frac{1}{2}(\ln(x))^2\right]_1^{e^3} \\
&= \frac{1}{2}[(\ln(e^3))^2 - (\ln(1))^2] \\
&= \frac{1}{2}(9-0) = \frac{9}{2}
\end{aligned}$$

Verify:

$$\frac{d}{dx}\left(\frac{(\ln(x))^2}{2}\right) = \frac{1}{2}\cdot(2)\cdot(\ln(x))\cdot\frac{1}{x} = \frac{\ln(x)}{x}.$$

2. $\displaystyle\int_1^{e^3} \frac{\ln(x)}{x^2}\,dx$

This integral looks remarkably similar to the first one. This time, however, when we write the integrand as a product

$$(\ln(x))\cdot(\frac{1}{x^2})$$

we see that we no longer have a function and its derivative. Since we have a product of two *unrelated* pieces, we try integration by parts.

Try:

$u = \ln(x)$ $dv = \frac{1}{x^2}\,dx$

$du = \dfrac{1}{x}\,dx$ $v = -\frac{1}{x}$

The notation for integration by parts is not really amenable to definite integrals, so we'll begin with the related indefinite integral. Using $\int udv = uv - \int vdu$, we get

$$\begin{aligned}\int \frac{\ln(x)}{x^2}\,dx &= uv - \int v\,du \\ &= -\frac{\ln(x)}{x} + \int \frac{1}{x^2}\,dx \\ &= -\frac{\ln(x)}{x} - \frac{1}{x} + C \\ &= -\frac{\ln(x)-1}{x} + C\end{aligned}$$

Still have to plug in limits, but first:

Verify:

$$\frac{d}{dx}\left(-\frac{\ln(x)}{x} - \frac{1}{x} + C\right) = \frac{x(\frac{-1}{x}) + \ln(x)(1)}{x^2} + \frac{1}{x^2} = \frac{-1+\ln(x)+1}{x^2} = \frac{\ln(x)}{x^2}.$$

Finishing the definite integral now:

$$\begin{aligned}\int_1^{e^3} \frac{\ln(x)}{x^2}\,dx &= -\frac{(\ln(e^3)-1)}{e^3} + \frac{(\ln(1)-1)}{1} \\ &= -\frac{3-1}{e^3} + \frac{-1}{1} \\ &= -\frac{2}{e^3} - 1\end{aligned}$$

3. $\displaystyle\int e^x \sin(x)\,dx$

Since the integrand is a product of two "building block" functions that are completely unrelated by differentiation, we know we're using integration by parts. Neither e^x and $\sin(x)$ become more complicated upon antidifferentiation, or less complicated upon differentiation, so our choice for u and dv is random.

Let

$u = e^x$ $dv = \sin(x)\ dx$

$du = e^x$ $v = -\cos(x)$

Using $\int u\ dv = uv - \int v\ du$, we get

$$\int e^x \sin(x)\ dx = -e^x \cos(x) + \int e^x \cos(x)\ dx$$

We have another integral containing a product of unrelated terms, so we use integration by parts. We're afraid, though, that we're just going in a circle.

$u = e^x$ $dv = \cos(x)\ dx$

$du = e^x$ $v = \sin(x)$

$$\int e^x \sin(x)\ dx = -e^x \cos(x) + e^x \sin(x) - \int_e^x \sin(x)\ dx.$$

At first, it looks like our fears of circular calculations have come true. But notice that the integral on the right is the same as the integral on the left. We can add that integral to both sides of the equations, giving us twice the integral on the left, and eliminating the integral on the right.

$$\begin{aligned} 2\int e^x \sin(x)\ dx &= -e^x \cos(x) + e^x \sin(x) \\ \int e^x \sin(x)\ dx &= \frac{e^x}{2}(\sin(x) - \cos(x)) \end{aligned}$$

4. $\int \sec(x)\ dx$. *Hint:* Consider multiplying by 1 in the form $\dfrac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}$.

$$\begin{aligned} \int \sec(x)\ dx &= \int \sec(x) \cdot \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}\ dx \\ &= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}\ dx \end{aligned}$$

Substitution:

Let $u = \sec(x) + \tan(x)$. Then $du = \sec(x)\tan(x) + \sec^2(x)\ dx$.

Substituting in for u and du, we get

$$\begin{aligned} \int \sec(x)\ dx &= \int \frac{1}{u}\ du \\ &= \ln|\sec(x) + \tan(x)| + C \end{aligned}$$

Verify:

$$\frac{d}{dx}(\ln|\sec(x)+\tan(x)|+C) = \frac{1}{\sec(x)+\tan(x)} \cdot (\sec(x)\tan(x)+\sec^2(x)) = \sec(x).$$

5. $\displaystyle\int x\sqrt{3-2x}\ dx$

When you look at this, you see a product of two seemingly unrelated terms, one of which is a composition. The product of seeminly unrelated terms says "integration by parts" to you, and the composition says "substitution" to you.

Let's try both methods:

Method 1: *Integration by parts:*

x differentiates away to just 1, and $\sqrt{3-2x}$ isn't hard to antidifferentiate with a little substitution, so we'll try those choices for u and dv:

Let

$u = x$	$dv = \sqrt{3-2x}\ dx$
$du = dx$	$v =$ (using a substitution with $w = 3 - 2x$),
	$\Rightarrow v = -\frac{1}{2} \cdot \frac{2}{3}(3-2x)^{3/2} = -\frac{1}{3}(3-2x)^{3/2}$

Using $\int u\ dv = uv - \int v\ du$, we find that with *this* choice of u and dv, we get:

$$\int x\sqrt{3-2x}\ dx = -\frac{x}{3}(3-2x)^{3/2} + \frac{1}{3}\int (3-2x)^{3/2}\ dx.$$

The integral on the right can be done using the same substitution we used before:

$$\begin{aligned}
\int x\sqrt{3-2x}\,dx &= -\frac{x}{3}(3-2x)^{3/2} - \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2}{5}(3-2x)^{5/2} + C \\
&= -\frac{x}{3}(3-2x)^{3/2} - \frac{1}{15}(3-2x)^{5/2} + C \\
&= -(3-2x)^{3/2}(\frac{x}{3} + \frac{1}{15}(3-2x)) + C \\
&= -(3-2x)^{3/2}(\frac{x}{3} + \frac{1}{5} - \frac{2x}{15}) + C \\
&= -(3-2x)^{3/2}(\frac{5x}{15} - \frac{2x}{15} + \frac{1}{5}) + C \\
&= -(3-2x)^{3/2}(\frac{3x}{15} + \frac{1}{5}) + C \\
&= -(3-2x)^{3/2}(\frac{x+1}{5}) + C \\
&= -\frac{1}{5}(x+1)(3-2x)^{3/2} + C
\end{aligned}$$

You'll notice I did *a lot* of algebraic simplification. Anything after the first line is perfectly acceptable; it's just that since I'm going to be doing this in two different ways, I want you to be able to see that the results are the same either way.

Method 2: *Substitution:*

The composition is $(3-2x)^{1/2}$, and the inside of that composition is $3-2x$, so that's what we'll let u be.

Let $u = 3-2x$. Then $du = -2\,dx$, so $-\frac{1}{2}\,du = dx$.

When I make the substitution, I realize I have a problem:

$$\int x\sqrt{3-2x}\,dx = -\frac{1}{2}\int x\cdot u^{1/2}\,du.$$

While oftentimes this sort of situation (mixed x's and u's) is irredeemable, this particular situation is an example of a sometime handy technique. We can actually solve for x in terms of u, in this particular situation, and that will allow us to get rid of that last x.

Since $u = 3 - 2x$, $-\frac{1}{2}(u-3) = x$. Thus we have

$$\int x\sqrt{3-2x}\,dx = \frac{1}{4}\int (u-3)u^{1/2}\,du.$$

If you compare the left side to the right side, you can see what we've done with our little substitution. Originally, we had a product of two terms: one was just a single x, while the other was the square root of a difference, where the difference involved a x term and a constant. This can not be simplified in any useful manner. Our substitution has moved the difference to outside the square root, and moved the simple degree 1 term inside the square root. This *can* be simplifed in a useful way:

$$\begin{aligned}
\int x\sqrt{3-2x}\,dx &= \frac{1}{4}\int u^{3/2} - 3u^{1/2}\,du \\
&= \frac{1}{4}(\frac{2}{5}u^{5/2} - 3\cdot\frac{2}{3}u^{3/2}) + C \\
&= \frac{1}{10}u^{5/2} - \frac{1}{2}u^{3/2} + C \\
&= \frac{1}{10}(3-2x)^{5/2} - \frac{1}{2}(3-2x)^{3/2} + C \\
&= (3-2x)^{3/2}(\frac{1}{10}(3-2x) - \frac{1}{2}) + C \\
&= (3-2x)^{3/2}(\frac{3}{10} - \frac{x}{5} - \frac{5}{10}) + C \\
&= (3-2x)^{3/2}[-(\frac{2}{10} + \frac{x}{5})] + C \\
&= -\frac{1}{5}(x+1)(3-2x)^{3/2}
\end{aligned}$$

Again, you'll notice I did a lot of algebraic simplification. If in method 1, and in method 2, I had stopped after a couple lines of simplification, it would not have been at all apparent that these two methods ended up with the same result. However, with the simplification, you can see that in fact they did.

That should be enough verification that the integral is correct. In case it is not, however:

Verify:

$$\begin{aligned}
\frac{d}{dx}(-\frac{1}{5}(x+1)(3-2x)^{3/2}+C) &= -\frac{1}{5}[(x+1)\frac{3}{2}(3-2x)^{1/2}\cdot(-2) \\
&\quad +1\cdot(3-2x)^{3/2}] \\
&= -\frac{1}{5}[-3(x+1)(3-2x)^{1/2}+(3-2x)^{3/2}] \\
&= -\frac{1}{5}(3-2x)^{1/2}[-3(x+1)+(3-2x)^{1}] \\
&= -\frac{1}{5}(3-2x)^{1/2}[-5x] \\
&= x\sqrt{(}3-2x)
\end{aligned}$$

6. $\displaystyle\int x^3\sin(x^2)\,dx$

This problem can be approached two different ways. I will describe them both.

Method 1:
The integrand is a product, and the two pieces seem unrelated. I am therefore going to use integration by parts.

I can't antidifferentiate $\sin(x^2)$, so my first thought is as follows:

Let

$u=\sin(x^2)$ $\qquad dv=x^3\,dx$

$du=2x\sin(x^2)\,dx$ $\qquad v=\displaystyle\int\frac{1}{4}x^4\,dx$

Using $\int u\,dv=uv-\int v\,du$, I get

$$\int x^3\sin(x^2)\,dx=\frac{x^4}{4}\sin(x^2)-\frac{2}{4}\int x^5\sin(x^2)\,dx.$$

The integral I am left with has a higher degree of x than the one I started out with. This is not a good sign! While true, the above attempt was not useful.

What then should I do? I still can't antidifferentiate $\sin(x^2)$. Why not? Because $\sin(x^2)$ is a composition. I need the piece that would have come from differentiating the inside! In other words, looking at my u and du in the above attempt, I need a multiple of x.

In the original integral, do I have an x? You betcha! I have 3 – $x^3 \sin(x^2) = x^2 \cdot x \sin(x^2)$!

So ...

...Let

$u = x^2$ $\qquad dv = x \sin(x^2)\, dx$

$du = 2x\, dx$ $\qquad v = \int x \sin(x^2)\, dx$

To find v, we need a substitution.
Let $w = x^2$. Then $dw = 2x\, dx$,
so $v = \frac{1}{2} \int \sin(w)\, dw = -\frac{1}{2} \cos(x^2)$

$$\int x^3 \sin(x^2)\, dx = -\frac{1}{2} x^2 \cos(x^2) + \int x \cos(x^2)\, dx.$$

The integral remaining is remarkably similar to the one I had to do to go from dv to v. Doing a similar substitution, but this time in my head, I find

$$\int x^3 \sin(x^2)\, dx = -\frac{1}{2} x^2 \cos(x^2) + \frac{1}{2} \sin(x^2) + C$$

Verify:

$$\begin{aligned} \frac{d}{dx}\left(-\frac{1}{2}x^2 \cos(x^2) + \frac{1}{2}\sin(x^2) + C\right) &= -\frac{1}{2}[x^2 \cdot -\sin(x^2) \cdot 2x + \cos(x^2) \cdot 2x] + x\cos(x^2) \\ &= x^3 \sin(x^2) - x\cos(x^2) + x\cos(x^2) \\ &= x^3 \sin(x^2). \end{aligned}$$

This first method involved me deciding right off that because the two pieces of the product are unrelated, this must be an integration by parts problem. I ended up having to do a substitution as well.

What if I had focused instead on the indisputable fact that $\sin(x^2)$ is a composition? In order for $x^3 \sin(x^2)$ to have come from differentiating something, the chain rule would undoubtedly have to have been used. So ... what if I had started with substitution? ...

Method 2:
When we're trying to antidifferentiate a composition, we let u be the inside function. So

Let $u = x^2$. Then $du = 2x\ dx$, and so $\frac{1}{2}\ du = x\ dx$.

How am I going to substitute these results into $x^3 \sin(x^2)\ dx$? If we don't recognize that $x^3 = x^2 \cdot x$, then another way is to solve for dx:

$$\frac{1}{2}\ du = x\ dx \Longrightarrow dx = \frac{1}{2x}\ du.$$

We then end up with

$$\begin{aligned}
\int x^3 \sin(x^2)\ dx &= \int x^3 \sin(u) \frac{1}{2x}\ du \\
&= \frac{1}{2} \int x^2 \sin(u)\ du \\
&= \frac{1}{2} \int u \sin(u)\ du
\end{aligned}$$

This is much simpler than what we started with, but now what? At this point, we recognize that we have a product of two unrelated pieces, and so we need to use integration by parts. I'll replace the u in the integration by parts formula with a U, to avoid confusion.

Let
$U = u$ $dv = \sin(u)\ du$
$dU = du$ $v = -\cos(u)$

Using $\int U\ dv = Uv - \int v\ dU$, we get

$$\begin{aligned}
\int x^3 \sin(x^2)\ dx &= \frac{1}{2} \int u \sin(u)\ du \\
&= \frac{1}{2}[-u\cos(u) + \int \cos(u)\ du] \\
&= \frac{1}{2}[-u\cos(u) + \sin(u)] + C \\
&= \frac{1}{2}[-x^2\cos(x^2) + \sin(x^2)] + C \\
&= -\frac{1}{2}x^2\cos(x^2) + \frac{1}{2}\sin(x^2) + C
\end{aligned}$$

So in the second method, I started with substitution, and then got to a point where I had to use integration by parts. In other words, the two methods are really the same, just approached in different orders, depending on what strikes you first when you're beginning the problem.

7. $\displaystyle\int \frac{\arcsin(x)}{\sqrt{1-x^2}}\,dx$

When we look at this, we certainly see a composition: $1-x^2$ is "plugged into" $\sqrt{x}$. However, we also recognize $\dfrac{1}{\sqrt{1-x^2}}$ as being the derivative of $\arcsin(x)$. If $\arcsin(x)$ weren't sitting inside our integral as well, we might just disregard that as coincidence. However, since $\arcsin(x)$ *is* sitting right there, we decide it probably is *not* merely coincidence.

Let $u = \arcsin(x)$. Then $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$, which is sitting right in our integral. Thus

$$\int \frac{\arcsin(x)}{\sqrt{1-x^2}}\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\arcsin(x))^2 + C.$$

Verify:

$$\frac{d}{dx}\left(\frac{1}{2}(\arcsin(x))^2 + C\right) = \frac{1}{2}\left(2\arcsin(x)\frac{1}{\sqrt{1-x^2}}\right) = \frac{\arcsin(x)}{\sqrt{1-x^2}}.$$

8. $\displaystyle\int \arctan(x)\,dx$

How in the world to approach this? $\arctan(x)$ is a "building block" function – doesn't that mean we either know how to integrate or we don't?

Furthermore, there's no composition involved, so we couldn't possibly use substitution!

That leaves us with only integration by parts, although we're at a loss at how to approach it (unless we remember $\int \ln(x)\,dx$ from yesterday).

There's only one (sensible) way to write $\arctan(x)$ as a product:

$$\arctan(x) = \arctan(x)\cdot 1.$$

I can't use $dv = \arctan(x)\ dx$, as then $v = \int \arctan(x)\ dx$, which is the problem I started with! So ...

Let $u = \arctan(x)$ $\qquad dv = dx$
$du = \frac{1}{1+x^2}\ dx$ $\qquad v = x$

Using the by now very familiar $\int u\ dv = uv - \int v\ du$, I find

$$\int \arctan(x)\ dx = x\arctan(x) - \int \frac{x}{1+x^2}\ dx.$$

This gives us another integral, but this one is a fairly straightforward substitution problem. With $u = 1+x^2$ and $du = 2x\ dx$ (so that $\frac{1}{2}\ du = x\ dx$), we find that

$$\int \arctan(x)\ dx = x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C$$

Verify:

$$\begin{aligned}
\frac{d}{dx}(x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C) &= [x \cdot \frac{1}{1+x^2} + 1 \cdot \arctan(x)] - \frac{1}{2} \cdot \frac{1}{1+x^2} \cdot 2x \\
&= \frac{x}{1+x^2} + \arctan(x) - \frac{x}{1+x^2} \\
&= \arctan(x)
\end{aligned}$$