Determine whether each of the following improper integrals converges or diverges.

1. $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}-2}\,dx$

   Our first choice would be to just evaluate this improper integral directly by first rewriting it as a limit and then using antidifferentiation, as we did Tuesday. However, we don't know how to antidifferentiate $\dfrac{1}{\sqrt{x}-2}$. On to plan B.

   Plan B is to compare $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}-2}\,dx$ to an improper integral whose behavior we know. At this point, we only know the behavior of $\displaystyle\int_a^\infty f(x)\,dx$ if $f(x)$ is of the form $\dfrac{1}{x^p}$.

As long as $a > 0$,	
$\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$	converges if $p > 1$
	diverges if $p \le 1$

   So we need to find some function $\dfrac{1}{x^p}$ to compare with $\dfrac{1}{\sqrt{x}-2}$.

   Especially in the beginning of your comparison experience, start with the simplest comparisons. That is, don't worry (at least at first) about whether the thing your comparing with has an improper integral that converges or diverges – you don't yet have much of an intuition for this anyway. Just find a true comparison and see whether it's helpful to you or not.

   In my experience, the easiest way to find something to compare with your fraction is to begin with the denominators. In this case, our denominator is $\sqrt{x}-2$. What is the *easiest* thing to compare $\sqrt{x}-2$ to? x^3, of course.

$$\begin{aligned}\sqrt{x}-2 &< \sqrt{x}\\ \frac{1}{\sqrt{x}-2} &> \frac{1}{\sqrt{x}}\\ \int_2^\infty \frac{1}{\sqrt{x}-2}\,dx &\geq \int_2^\infty \frac{1}{\sqrt{x}}\,dx\end{aligned}$$

Since $\frac{1}{\sqrt{x}}$ is of the form $\frac{1}{x^p}$, with $p < 1$ *and* since we're integrating from 2 to ∞, the integral on the right diverges. Remember, that means that $\int_2^\infty \frac{1}{\sqrt{x}}\,dx$ is infinite.

Since the integral on the left is bigger than the one on the right, which is heading off to infinity, the integral on the left must also be heading off to infinity as well. In other words,

$$\int_2^\infty \sqrt{x}-2\,dx \text{ diverges.}$$

Thus the comparison we found was helpful.

2. $\int_2^\infty \frac{4}{x^3+2}\,dx$

Again, our first choice would be to rewrite this as a limit and then use the Fundamental Theorem of Calculus. However, as with the first problem, we don't know *how* to antidifferentiate $\frac{4}{x^3+2}$. Again, we switch to simply determining convergence or divergence, using the comparison theorem.

$$\begin{aligned}x^3+2 &> x^3\\ 0 < \frac{4}{x^3+2} &< \frac{4}{x^3}\\ 0 \leq \int_2^\infty \frac{4}{x^3+2}\,dx &\leq \int_2^\infty \frac{4}{x^3}\,dx\end{aligned}$$

The integral on the right is just $4\int_2^\infty \frac{1}{x^3}\,dx$. Since $\frac{1}{x^3}$ is of the form $\frac{1}{x^p}$, with $p > 1$ *and* since we're integrating from 2 to ∞, the integral on the right converges. Remember, that means that it's a finite number.

Thus we have that

$$0 \le \int_2^\infty \frac{4}{x^3+2}\,dx \le \text{ a finite number},$$

so $\int_2^\infty \frac{4}{x^3+2}\,dx$ is a finite number as well. In other words,

$$\int_2^\infty \frac{4}{x^3+2}\,dx \text{ converges.}$$

Again, going with the simplest comparison proved useful.

> **Be aware:** The easiest comparisons don't always prove useful. Suppose that instead of $\int_2^\infty \frac{1}{x^3+2}\,dx$, we'd been dealing instead with $\int_2^\infty \frac{1}{x^3-2}\,dx$. That small change from addition to subtraction makes this problem considerably more tricky. See the end of this document for a discussion of what the problem is, and one way to deal with it.

3. $\displaystyle\int_0^1 \frac{2}{\sqrt{x}+x^2}\,dx$

$$\begin{aligned}
\sqrt{x}+x^2 &> \sqrt{x} \text{ and } x^2 \\
\frac{2}{\sqrt{x}+x^2} &< \frac{1}{\sqrt{x}} \text{ and } \frac{1}{x^2} \\
\int_0^1 \frac{2}{\sqrt{x}+x^2}\,dx &\le \int_0^1 \frac{1}{\sqrt{x}}\,dx \text{ and } \int_0^1 \frac{1}{x^2}\,dx
\end{aligned}$$

The second integral on the right diverges. Being less than or equal to infinity is not a useful comparison. So that's *not helpful!*

But the first integral converges, which is helpful!

Thus $\int_0^1 \frac{2}{\sqrt{x}+x^2}\,dx$ converges as well, by comparison to $\int_0^1 \frac{1}{\sqrt{x}}\,dx$.

4. $\displaystyle\int_0^\infty \frac{2}{\sqrt{x}+x^2}\,dx$

 This is improper at both ends, so we need to write it like:

$$\int_0^\infty \frac{2}{\sqrt{x}+x^2}\,dx = \int_0^1 \frac{2}{\sqrt{x}+x^2}\,dx + \int_1^\infty \frac{2}{\sqrt{x}+x^2}\,dx$$

 We already know the first integral on the right converges – we just did that.

 How about the first one?

$$\begin{aligned}\sqrt{x}+x^2 &> \sqrt{x} \text{ and } x^2\\ \frac{2}{\sqrt{x}+x^2} &< \frac{1}{\sqrt{x}} \text{ and } \frac{1}{x^2}\\ \int_1^\infty \frac{2}{\sqrt{x}+x^2}\,dx &\leq \int_1^\infty \frac{1}{\sqrt{x}}\,dx \text{ and } \int_1^\infty \frac{1}{x^2}\,dx\end{aligned}$$

 In this case, the first integral on the right diverges, and so that's a true but useless comparison; but the second integral on the right converges, and so the original converges.

 Putting it all together, we're adding up two finite pieces, and so the whole thing also converges.

Dealing with slightly more difficult situations:

Suppose that instead of $\int \frac{1}{x^3+2}\,dx$, as we had in the second problem, we have $\int \frac{1}{x^3-2}\,dx$. How does that change things?

Consider the obvious comparison:

$$\begin{aligned}x^3-2 &< x^3\\ \frac{1}{x^3-2} &> \frac{1}{x^3}\\ \int_2^\infty \frac{1}{x^3-2}\,dx &\geq \int_2^\infty \frac{1}{x^3}\,dx\end{aligned}$$

The improper integral on the right converges – that is, it's finite. So we have

$$\int_2^\infty \frac{1}{x^3-2}\,dx \geq \text{ a finite number.}$$

This is *not* a useful comparison. We can not deduce from this inequality whether $\int_2^\infty \frac{1}{x^3-2}\,dx$ is finite or whether in fact it's infinite.

In this case, we have to try some other comparison. We still want the inequality to be obviously true. Up to now, we've just dropped the constant that's been added or subtracted. But in this problem that didn't work.

Can I try *replacing* the constant with something useful? In other words, I want to be able to say either

$$x^3 - 2 < x^3 - \text{something useful} \qquad \text{or} \qquad x^3 - 2 > x^3 - \text{something useful.}$$

In order to be useful, it must combine nicely with what I've already got – my x^3 term. Otherwise, I'm stuck with a worse mess in the denominator than the $x^3 - 2$ I've got now.

The only things that combine nicely with x^3 are multiples of x^3. So I want to be able to say

$$x^3 - 2 < x^3 - ax^3 \qquad \text{or} \qquad x^3 - 2 > x^3 - ax^3.$$

Remember, eventually I'm going to be taking the reciprocal of these and integrating them. But also remember, the comparison theorem only applies to positive functions. That means that my comparison, $x^3 - ax^3$, must be positive, which means that a must be less than 1.

So, can I think of something true to say about some fraction of x^3 versus 2 on the interval $[2, \infty]$?

On $[2, \infty]$, $x^3 > 8$, so $\frac{1}{2}x^3 > 4$, which means that

$$\begin{aligned}
\frac{1}{2}x^3 &> 2 \\
\Rightarrow 2 &< \frac{1}{2}x^3 = \frac{x^3}{2} \\
\Rightarrow x^3 - 2 &> x^3 - \frac{x^3}{2} \\
\Rightarrow x^3 - 2 &> \frac{x^3}{2} \\
\Rightarrow \frac{1}{x^3 - 2} &< \frac{2}{x^3} \\
\Rightarrow \int_2^\infty \frac{1}{x^3 - 2}\, dx &\leq 2\int_2^\infty \frac{1}{x^3}\, dx
\end{aligned}$$

The integral on the right is finite, since $p = 3$, which is greater than 1 (and since we're still looking at intervals that don't include 0). That means that the integral on the left is smaller than a finite number, but is positive. Hence

$$\int_2^\infty \frac{1}{x^3 - 2}\, dx \text{ converges.}$$