Goals:

Goal: Figure out what to do when faced with an improper integral that we can't evaluate simply by taking limits and antidifferentiating.

1. Is there any way to at least determine whether or not an improper integral I converges even if we cannot find an antiderivative?

2. Better yet, if we *do* determine that an improper integral I converges, is there a way to approximate the value of the integral I?

Let $I = \int_a^\infty f(x)\, dx$.

Dealing with Goal 1:

1. If $f(x)$ **is** antidifferentiable, cope with I by taking the limit of proper definite integrals. This tells us whether I diverges or converges, and if so, what it converges to.

2. If $f(x)$ is **not** antidifferentiable, then we try to determine whether or not I converges by comparing it to an improper integral whose convergence or divergence we know:

   (a) If I is *less* than or equal to a convergent improper integral (but greater than or equal to 0), it must converge also. If it is *greater than* a convergent improper integral, our comparison was useless.

   (b) If I is *greater* than or equal to a (positive) divergent improper integral, then it must diverge also. If it is *less than* a divergent improper integral, our comparison was useless.

Still left to figure out: Goal 2

If the integrand of an improper integral is *not* antidifferentiable, and you've already determined the improper integral converges, how can you approximate what it converges to?

Plan for approximating a convergent improper integral:

1. **Replace the improper integral with a proper one**

   Replace $\int_a^\infty f(x)\,dx$ with $\int_a^t f(x)\,dx$.

   Since these are not equal, this replacement will result in an error. The error is the *tail*, $\int_t^\infty f(x)\,dx$.

   Because the bigger t is, the smaller the tail is, we can control the error introduced by this replacement by making t sufficiently large.

2. **Approximate the proper integral with one of our usual techniques**

   Approximate $\int_a^t f(x)\,dx$ using left, right, midpoint, or trapezoidal sums. This will of course introduce another error. We can control the error introduced by using a Riemann Sum using our error bound formulae from Section 6.2.

3. **Bound the total error**

   The total error will be the sum of the two errors,

$$\text{total error} = \text{tail} + \text{error using Riemann sum.}$$