Find $\int x^2 \arctan(x)\, dx$.

Choose

$u = x^2 \arctan(x)$ $\qquad dv = dx$

$du = 2x \arctan(x) + \dfrac{x^2}{1+x^2}\, dx$ $\qquad v = x$

Using $\int u dv = uv - \int v du$, we get

$$\begin{aligned}\int x^2 \arctan(x)\, dx &= x^3 \arctan(x) - \int 2x^2 \arctan(x) + \frac{x^3}{1+x^2}\, dx \\ &= x^3 \arctan(x) - 2\int x^2 \arctan(x)\, dx - \int \frac{x^3}{1+x^2}\, dx\end{aligned}$$

Adding $2\int x^2 \arctan(x)\, dx$ to both sides, we get

$$\begin{aligned}3\int x^2 \arctan(x)\, dx &= x^3 \arctan(x) - \int \frac{x^3}{1+x^2}\, dx \\ \int x^2 \arctan(x)\, dx &= \frac{1}{3}x^3 \arctan(x) - \frac{1}{3}\int \frac{x^3}{1+x^2}\, dx\end{aligned}$$

Now all we need to do that last not-very-hard looking integral. If you know polynomial long-division-or are willing to learn it (it's not that hard)- this integral really *isn't* very hard to do:

First, it's easier to see what's going on if we rewrite both the numerator and the denominator of our fraction $\dfrac{x^3}{1+x^2}$ so that the terms are listed in order of descending degree: $\dfrac{x^3}{x^2+1}$.

$$x^2+1 \;\overline{)\,x^3}$$

How many times does x^2+1 go into x^3? Well, I'd have to multiply the x^2 term by x in order to get x^3 – unlike in other division, we don't worry about the lower order terms.

Thus we get
$$\begin{array}{rll} & x & \\ x^2+1 & \overline{)\,x^3} & \\ & -\underline{(x^3} & \underline{+x)} \\ & & -x \end{array}$$

Now we ask ourselves how many times x^2+1 goes into x. Well, there isn't any non-negative power of x that we could multiply the x^2 term by to get x, so we have a remainder of $-x$.

Thus

$$\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}.$$

We can check that this is true by multiplying both the left and right sides by the denominator x^2+1:

$$x^3 = x(x^2+1) - x \Longrightarrow x^3 = x^3 + x - x \Longrightarrow x^3 = x^3,$$

which is clearly true.

Thus we have that

$$\begin{aligned} \int x^2 \arctan(x)\,dx &= \frac{1}{3}x^3 \arctan(x) - \frac{1}{3}\int x - \frac{x}{1+x^2}\,dx \\ &= \frac{1}{3}\arctan(x) - \frac{1}{3}(\frac{1}{2}x^2 - \text{something easily done using } u\text{-substitution}). \end{aligned}$$

If we let $u = 1+x^2$, then $du = 2x\,dx$, so $\frac{1}{2}\,du = x\,dx$, so

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln(1+x^2).$$

Thus

$$\int x^2 \arctan(x)\,dx = \frac{1}{3}x^3 \arctan(x) - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + C.$$

Of course, to finish the problem, you have to plug in the limits of integration, and then check using the midpoint method.

If you don't know how to polynomial long-division and are unwilling to learn it, the integral becomes a bit trickier.

Find $\int \frac{x^3}{1+x^2}\,dx$:

Rewrite this integral as

$$\int \frac{x^3}{1+x^2}\,dx = \int x^2 \frac{x}{1+x^2}\,dx.$$

Use integration by parts, with

$$\begin{array}{ll} u = x^2 & dv = \dfrac{x}{1+x^2}\,dx \\ du = 2x\,dx & v = \int \dfrac{x}{1+x^2}\,dx \\ & w = 1+x^2 \Rightarrow dw = 2x\,dx \Rightarrow \dfrac{1}{2}\,dw = x\,dx \\ & v = \dfrac{1}{2}\int \dfrac{1}{u}\,du = \dfrac{1}{2}\ln(u) = \dfrac{1}{2}\ln(1+x^2) \end{array}$$

we get

$$\begin{aligned} \int \frac{x^3}{1+x^2}\,dx &= x^2\left(\frac{1}{2}\ln(1+x^2)\right) - 2\cdot\frac{1}{2}\int x\ln(1+x^2)\,dx \\ &= \frac{1}{2}x^2\ln(1+x^2) - \int x\ln(1+x^2)\,dx \end{aligned}$$

In that last integral on the right, we have a composition here; let's let $u = 1+x^2$, in which case $du = 2x\,dx$, so $\frac{1}{2}\,du = x\,dx$. Then our integral becomes

$$\int x\ln(1+x^2)\,dx = \frac{1}{2}\int \ln(u)\,du.$$

Integrating $\ln(x)$, or in this case $\ln(u)$ is an integration by parts problem we've done before. In the interests of space and time, I will leap ahead to the result,

$$\begin{aligned} \int x\ln(1+x^2)\,dx &= \frac{1}{2}(u\ln(u) - u) \\ &= \frac{1}{2}((1+x^2)\ln(1+x^2) - (1+x^2)) \\ &= \frac{1}{2}(1+x^2)\ln(1+x^2) - \frac{1}{2}(1+x^2) \\ &= \frac{1}{2}\ln(1+x^2) + \frac{1}{2}x^2\ln(1+x^2) - \frac{1}{2} - \frac{1}{2}x^2 \end{aligned}$$

Thus

$$\begin{aligned}\int \frac{x^3}{1+x^2}\,dx &= \frac{1}{2}x^2\ln(1+x^2) - \frac{1}{2}\ln(1+x^2) - \frac{1}{2}x^2\ln(1+x^2) + \frac{1}{2} + \frac{1}{2}x^2 \\ &= -\frac{1}{2}\ln(1+x^2) + \frac{1}{2} + \frac{1}{2}x^2\end{aligned}$$

and so

$$\begin{aligned}\int x^2\arctan(x)\,dx &= \frac{1}{3}x^3\arctan(x) - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx \\ &= \frac{1}{3}x^3\arctan(x) - \frac{1}{3}(-\frac{1}{2}\ln(1+x^2) + \frac{1}{2} + \frac{1}{2}x^2) \\ &= \frac{1}{3}x^3\arctan(x) + \frac{1}{6}\ln(1+x^2) - \frac{1}{6} - \frac{1}{6}x^2 + C\end{aligned}$$

If you compare the result we just got with the one I found (the easier way) way back toward the bottom of the second page, you'll see that they differ by a constant – but that's okay, because antiderivatives can differ by a constant.

As you can see, I think, the effort it takes to learn polynomial long-division (which is at least sometimes taught in high school algebra anyway) is much better spent, I think, than the effort it takes to do this integral the other way: once you know polynomial long-division, you can use it on other integrals as well, like the one we saw today in class $\int \frac{x^2}{1+x^2}\,dx$ (almost but not quite the same as the one we just did).