For the vector-valued function $\overrightarrow{\mathbf{r}}(t) =< 4\cos(\pi t), 4\sin(\pi t), t >$, find

1. the unit tangent vector $\overrightarrow{\mathbf{T}}(t)$ at $t = 0$

$$\overrightarrow{\mathbf{T}}(t) = \frac{\overrightarrow{\mathbf{r}}\,'(t)}{\|\overrightarrow{\mathbf{r}}\,'(t)\|} = \frac{< -4\pi\sin(\pi t), 4\pi\cos(\pi t), 1 >}{\sqrt{16\pi^2+1}}$$

$$\Longrightarrow \overrightarrow{\mathbf{T}}(0) = \frac{< 0, 4\pi, 1 >}{\sqrt{16\pi^2+1}}$$

For the vector-valued function $\overrightarrow{\mathbf{r}}(t) = < 4\cos(\pi t), 4\sin(\pi t), t >$, find

2. the principal unit normal vector $\overrightarrow{\mathbf{N}}(t)$ at $t = 0$

$$\overrightarrow{\mathbf{N}}(t) = \frac{\overrightarrow{\mathbf{T}}\,'(t)}{\|\overrightarrow{\mathbf{T}}\,'(t)\|}, \qquad \overrightarrow{\mathbf{T}}(t) = \frac{< -4\pi\sin(\pi t), 4\pi\cos(\pi t), 1 >}{\sqrt{16\pi^2 + 1}}$$

$$\Longrightarrow \overrightarrow{\mathbf{N}}(t) = \frac{\frac{1}{\sqrt{16\pi^2+1}} < -4\pi^2\cos(\pi t), -4\pi^2\sin(\pi t), 0 >}{\frac{1}{\sqrt{16\pi^2+1}}(\sqrt{(4\pi)^2})}$$

$$= \frac{1}{4\pi^2} < -4\pi^2\cos(\pi t), -4\pi^2\sin(\pi t), 0 >$$

$$= < -\cos(\pi t), -\sin(\pi t), 0 >$$

$$\overrightarrow{\mathbf{N}}(0) = < -1, 0, 0 >$$

3. Check to make sure $\overrightarrow{\mathbf{T}}(0)$ and $\overrightarrow{\mathbf{N}}(0)$ are orthogonal and have length 1.

We found:

$$\overrightarrow{\mathbf{T}}(0) = \frac{< 0, 4\pi, 1 >}{\sqrt{16\pi^2 + 1}} \qquad \overrightarrow{\mathbf{N}}(0) =< -1, 0, 0 > .$$

$$\frac{< 0, 4\pi, 1 >}{\sqrt{16\pi^2 + 1}} \cdot < -1, 0, 0 >= 0 + 0 + 0 = 0.$$

$$\begin{aligned} \|\overrightarrow{\mathbf{T}}(0)\| &= \frac{1}{\sqrt{16\pi^2 + 1}} \left(\sqrt{0^2 + (4\pi)^2 + 1^2} \right) = 1 \\ \|\overrightarrow{\mathbf{N}}(0)\| &= \sqrt{(-1)^2 + 0^2 + 0^2} = 1 \end{aligned}$$

4. For the vector-valued function $\overrightarrow{\mathbf{r}}(t) =< 4\cos(\pi t), 4\sin(\pi t), t >$, find the tangential and normal components of acceleration

Finding a_T, Method 1:

Using that $a_T = \dfrac{d^2s}{dt^2} = \dfrac{d}{dt}\left(\|\overrightarrow{\mathbf{r}}'(t)\|\right)$:

$$
\begin{aligned}
a_T &= \frac{d}{dt}\left(\|\overrightarrow{\mathbf{r}}'(t)\|\right) \\
&= \frac{d}{dt}\left(\| < -4\pi\sin(\pi t), 4\pi\cos(\pi t), 1 > \|\right) \\
&= \frac{d}{dt}\sqrt{16\pi^2\sin^2(\pi t) + 16\pi^2\cos^2(\pi t) + 1} \\
&= \frac{d}{dt}\sqrt{16\pi^2 + 1} \\
&= 0
\end{aligned}
$$

4. For the vector-valued function $\overrightarrow{\mathbf{r}}(t) =< 4\cos(\pi t), 4\sin(\pi t), t >$, find the tangential and normal components of acceleration

Finding a_T, Method 2:

Using that a_T is the component of $\overrightarrow{\mathbf{a}}(t)$ in the direction of $\overrightarrow{\mathbf{T}}(t)$:

$$\begin{aligned}
a_T &= \text{comp}_{\overrightarrow{\mathbf{T}}(t)} \overrightarrow{\mathbf{a}}(t) = \overrightarrow{\mathbf{a}}(t) \cdot \frac{\overrightarrow{\mathbf{T}}(t)}{\|\overrightarrow{\mathbf{T}}(t)\|} = \overrightarrow{\mathbf{a}}(t) \cdot \overrightarrow{\mathbf{T}}(t) \\
&= \overrightarrow{\mathbf{r}}''(t) \cdot \frac{\overrightarrow{\mathbf{r}}'(t)}{\|\overrightarrow{\mathbf{r}}'(t)\|} \\
&= \frac{1}{\sqrt{16\pi^2+1}} < -16\pi^2\cos(\pi t), -16\pi^2\sin(\pi t), 0 > \\
&\qquad \cdot < -4\pi\sin(\pi t), 4\pi\cos(\pi t), 1 > \\
&= \frac{1}{\sqrt{16\pi^2+1}}(64\pi^3\sin(\pi t)\cos(\pi t) - 64\pi^3\sin(\pi t)\cos(\pi t) + 0) \\
&= 0
\end{aligned}$$

No matter which method we used to find a_T, we found that the component of the acceleration in the tangent vector's direction is 0. Does this make sense?

Since we just found that for this vector-valued function the acceleration is orthogonal to the tangent vector (remember, we found using Method 2 that $\overrightarrow{\mathbf{a}}(t) \cdot \overrightarrow{\mathbf{T}}(t) = 0$), we wouldn't expect the acceleration to have any component in the direction of the tangent vector. It's all in the direction of the normal vector!

This makes find a_N easier than usual. I will still go through the steps of 2 methods, to help illustrate the ideas.

4. For the vector-valued function $\overrightarrow{\mathbf{r}}(t) =< 4\cos(\pi t), 4\sin(\pi t), t >$, find the tangential and normal components of acceleration

Finding a_N, Method 1:

Because $\overrightarrow{\mathbf{a}}(t) = a_T\overrightarrow{\mathbf{T}}(t) + a_N\overrightarrow{\mathbf{N}}(t)$ represents writing $\overrightarrow{\mathbf{a}}(t)$ as a sum of orthogonal vectors, we know that $\overrightarrow{\mathbf{a}}(t)$ represents the hypotenuse of a right triangle, while the other two vectors are the legs.
That is,

$$\begin{aligned} \|\overrightarrow{\mathbf{a}}(t)\|^2 &= \|a_T\overrightarrow{\mathbf{T}}(t)\|^2 + \|a_N\overrightarrow{\mathbf{N}}(t)\|^2 = (a_T)^2\,\|\overrightarrow{\mathbf{T}}(t)\|^2 + (a_N)^2\,\|\overrightarrow{\mathbf{N}}(t)\|^2 \\ &= (a_T)^2 + (a_N)^2 . \end{aligned}$$

Since $\overrightarrow{\mathbf{a}}(t) =< -16\pi^2\cos(\pi t), -16\pi^2\sin(\pi t), 0 >$, $\|\overrightarrow{\mathbf{a}}(t)\| = 16\pi^2$.
We also know that $a_T = 0$. Thus

$$16\pi^2 = 0^2 + {a_N}^2 \Longrightarrow a_N = 16\pi^2.$$

(Remember, $a_N > 0$ because $a_N = \kappa\left(\dfrac{ds}{dt}\right)^2 > 0$.)

4. For the vector-valued function $\overrightarrow{\mathbf{r}}(t) =< 4\cos(\pi t), 4\sin(\pi t), t >$, find the tangential and normal components of acceleration

Finding a_N, Method 2:

$$\overrightarrow{\mathbf{a}}(t) = a_T\overrightarrow{\mathbf{T}}(t) + a_N\overrightarrow{\mathbf{N}}(t) \qquad \Longrightarrow \qquad a_N\overrightarrow{\mathbf{N}}(t) = \overrightarrow{\mathbf{a}}(t) - a_T\overrightarrow{\mathbf{T}}(t).$$

In (2) and in 4(a), we found that

$$\begin{aligned}\overrightarrow{\mathbf{N}}(t) &= < -\cos(\pi t), -\sin(\pi t), 0 > \\ a_T &= 0 \\ \overrightarrow{\mathbf{a}}(t) &= < -16\pi^2\cos(\pi t), -16\pi^2\sin(\pi t), 0 >\end{aligned}$$

$$a_N < -\cos(\pi t), -\sin(\pi t), 0 >= 16\pi^2 < -\cos(\pi t), -\sin(\pi t), 0 > -\overrightarrow{\mathbf{0}}.$$

Equating x-components, y-components, and z-components

$$-a_N\cos(\pi t) = -16\pi^2\cos(\pi t),\; -a_N\sin(\pi t) = -16\pi^2\sin(\pi t),\; 0 = a_N(0).$$

Solving either of the first two eqns gives $a_N = 16\pi^2$.